Uniform Error Estimates for Finite Differences Methods Applied to Fluid Motion with Interfaces

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Finite differences with regular grids, irregular boundaries with jump conditions at the bdry or interface especially fluid flow with viscosity, Navier-Stokes eq’ns

Estimates in maximum norm for several equations
(1) Discrete Poisson equation, gain in regularity
(2) Application to the immersed interface method
(3) Discrete diffusion equation
(4) Error estimates for Navier-Stokes with a moving interface, velocity and pressure, neglecting the error in interface position validating the general approach in a simple setting
The Problem and the Result

Navier-Stokes eq’ns in $\mathbb{R}^2$ or $\mathbb{R}^3$
periodic boundary conditions
interface $\Gamma(t)$, a closed surface
force $f$ on $\Gamma$ acts on fluid
$\text{Re} = O(1)$

\[ \frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p + \mu \Delta u + f \delta_\Gamma \]
\[ \nabla \cdot u = 0 \]

The interfacial force $f$ amounts to jumps in $\nabla u$ and $p$.

**Result.** We assume $\Gamma(t)$ known; $f$, $\nabla_{\text{tan}} f$ known to $O(h^2)$. With one scheme based on the immersed interface method, with truncation error $O(h)$ at $\Gamma$ and $O(h^2)$ elsewhere, the error in velocity and pressure is uniformly about $O(h^2)$. 
Estimates in maximum norm for finite difference versions of equations such as

\[ \Delta u = f \]
\[ u_t = \Delta u + f \]
on a rectangular grid in \( \mathbb{R}^d \), any dimension \( d \)

Estimates in \( L^2 \) are the most natural.

For problems with limited smoothness, e.g. with interfaces, largest error is typically on a small set

\( L^\infty \) is a more meaningful measure of error than \( L^2 \)

methods are designed to control maximum truncation error

We get estimates of the form \( \Delta_h u^h = f^h \to \| u^h \| \leq C |f^h| \)

Equations are linear, so that estimates apply to errors:
If \( \Delta_h u^h = f^h \) and \( \Delta_h u^\text{exact} = f^h + \varepsilon^h \), \( \varepsilon^h \) = truncation error,
we subtract to get \( \| u^h - u^\text{exact} \| \leq C |\varepsilon^h| \)
Discrete Laplacian \( (d = 2) \) with spacing \( h \):
\[
\Delta_h u(ih, jh) = \left( u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} \right) / h^2
\]
\[
\Delta_h u = D_1^+ D_1^- u + D_2^+ D_2^- u
\]
\[
D_1^+ u = (u_{i+1,j} - u_{i,j}) / h, \quad D_1^- u = (u_{i,j} - u_{i-1,j}) / h
\]
For smooth \( u \), \( \Delta_h u = \Delta u + O(h^2) \) from Taylor expansion
Does inverting \( \Delta_h \) gain differences?

For the exact problem \( \Delta u = f \), periodic, average zero
we have sharp \( L^p \) estimates, \( 1 < p < \infty \)
\[
\| D^k u \|_{L^p} \leq C_p \| f \|_{L^p}, \quad k \leq 2
\]
There are also “sharp” estimates in Hölder norms \( C^{k+\lambda} \)
(Schauder estimates) but \textbf{not} \( L^\infty \)
**Estimate for the Discrete Laplacian.** Suppose $u_h$ and $f_h$ are periodic grid functions on $\mathbb{R}^d$ with average zero and $\Delta_h u_h = f_h$. Then in maximum norms

$$\| u_h \| + \| D_h u_h \| \leq C \| f_h \|,$$

$$\| D_h^2 u_h \| \leq C \| \log h \| \| f_h \|$$

with $C$ independent of $h$, $D_h =$ any first difference.

The log factor cannot be improved.

E.g., $u(x, y) = (x^2 - y^2) \log r$

This can be proved using estimates for a discrete Green’s function. J.T.B. and A. Layton, CAMCoS (2006)

The discrete maximum principle estimates $u_h$, but not differences.

V. Thomée proved discrete Schauder estimates (1968).

Estimates for finite element spaces are better known.
Poisson problem with an interface

\[
\begin{align*}
\Delta u_- &= f_- \quad \text{in } \Omega_- , \\
\Delta u_+ &= f_+ \quad \text{in } \Omega_+ , \\
[u] &= g_0 \quad \text{on } \Gamma , \\
[\partial_n u] &= g_1 \quad \text{on } \Gamma
\end{align*}
\]
Poisson problem with an interface

\[ \Delta u_- = f_- \text{ in } \Omega_-, \quad \Delta u_+ = f_+ \text{ in } \Omega_+, \]

\[ [u] = g_0 \text{ on } \Gamma, \quad [\partial_n u] = g_1 \text{ on } \Gamma \]

\[ \Delta_h u(ih) - \Delta u(ih) = O(h^2) \]

at regular points,\n
away from \( \Gamma \)\n
\[ \Delta_h u(ih) - \Delta u(ih) \text{ is large} \]

at irregular points,\n
where \( \Delta_h \) crosses \( \Gamma \)

If we improve the truncation error to \( O(h) \) at the irreg. pts.\n
using the Immersed Interface Method, then\n
the error in \( u \) is uniformly \( O(h^2) \), and\n
the error in \( D_h u \) is uniformly \( O(h^2|\log h|) \)
Poisson problem with interface:

\[ \Delta u_- = f_- \quad \text{in } \Omega_- , \quad \Delta u_+ = f_+ \quad \text{in } \Omega_+ , \]

\[ [u] = g_0 \quad \text{on } \Gamma , \quad [\partial_n u] = g_1 \quad \text{on } \Gamma \]

Find \( C_h \) so that \( \Delta_h u_{\text{exact}} - f_{\text{exact}} = C_h + O(h) \). Solve

\[ \Delta_h u_h = f_h + C_h \]

\( C_h = 0 \) at regular grid points; truncation error \( O(h^2) \)
\( C_h \neq 0 \) at irregular grid points near \( \Gamma \), e.g.

If \( x_j \in \Omega_- , \ x_{j+1} \in \Omega_+ , \ h_+ = x_{j+1} - x^* \)

\[ v_{xx}(x_j) = \frac{v_{j-1} - 2v_j + v_{j+1}}{h^2} - \frac{1}{h^2} \left( [v] + h_+ [Dv] + \frac{h_+^2}{2} [D^2v] \right) + O(h) \]

Zhilin Li & K. Ito, The Immersed Interface Method..., SIAM, 2006
With corrections, \( \Delta_h(u_h - u) = O(h) \) at the irregular points since the \( C_h \)'s cancel. It is \( O(h^2) \) at the regular points.

**Theorem.** With truncation error \( O(h^2) \) at regular points and \( O(h) \) truncation error at irregular points,

\[
|u_h(jh) - u(jh)| \leq C_0 h^2, \\
|D_h u_h(jh) - D_h u(jh)| \leq C_1 h^2 |\log h|.
\]

Brief summary of proof (J.T.B. and A. Layton, CAMCoS '06): The \( O(h) \) truncation error at irregular points has the form

\[
F_0 + D_h^{(1)} F_1 + \cdots + D_h^{(d)} F_d, \quad F_i = O(h^2)
\]

Then \( \Delta_h(u_h - u) = F_{\text{reg}} + F_0 + \nabla_h \cdot (F_1, \ldots, F_d) \)

The estimate for \( u_h - u \), has a gain of one deriv for \( D_h(u_h - u) \), gain of two derivs, log factor
Lemma. Suppose $f^{irr}$ is a grid function on $\Omega$ such that $f^{irr} \neq 0$ only at irregular points (near the interface). Then there are periodic grid functions $F_k$ so that $f^{irr} = F_0 + \sum_{k=1}^d D_k^+ F_k$ and

$$\|F_k\|_{\text{max}} \leq Ch \|f^{irr}\|_{\text{max}}.$$ 

Key example in one dimension:
$F(ih) = 0$ for $i \leq 0$; $F(ih) = 1$ for $i > 0$
Set $D_h^+ F(ih) = [F((i + 1)h) - F(ih)]/h$
Then $D_h^+ F(ih) = 1/h$ for $i = 0$; $D_h^+ F(ih) = 0$ for $i \neq 0$
$D_h^+ F \to F$ gains a factor of $h$ in maximum norm.

To prove in dimension $d$, write $f^{irr}$ locally
as the difference of its sum in a direction
transverse to $\Gamma$; use partition of unity
Similar lemmas in $L^2$: Hackbusch ’81, Stevenson ’91
Gain in Regularity for Parabolic Difference Equations

Suppose we approximate (using backward Euler)

\[ u_t = \Delta u + f, \quad u(\cdot, 0) = 0 \]

by

\[ u^{n+1} - u^n = \tau \Delta_h u^{n+1} + \tau f^{n+1} \]

or

\[ u^{n+1} = (I - \tau \Delta_h)^{-1} (u^n + \tau f^{n+1}) \]

with

\[ t_n = n\tau, \quad \tau = \text{time step} = ch \]

Then

\[ \| u^n \| + \| D_h u^n \| \leq C_1 \sup_{t \leq T} \| f(\cdot, t) \| \]

\[ \| D_h^2 u^n \| \leq C_2 (1 + |\log h|) \sup_{t \leq T} \| f(\cdot, t) \| \]

Similar statements hold for a class of time-stepping methods...

Interpretation: \( u^{\text{computed}} - u^{\text{exact}} \) and differences are bounded by maximum truncation error.

Gain in Regularity for Diffusive Difference Equations

Discretize $u_t = \Delta u$ in $\mathbb{R}^d \times [0, T]$  
grid spacing $h$, time step $\tau$ 
use $\Delta_h$, usual second-order Laplacian, $\Delta_h = \sum_{\nu=1}^{d} D_{\nu}^+ D_{\nu}^-$ 
use implicit time stepping, $\tau = O(h)$ 
for a single step method, $u^{n+1} = S u^n$ and $u^n = S^n u^0$ 
Crank-Nicolson method (second-order accurate):  
\[ u^{n+1} - u^n = \left( \tau/2 \right) (\Delta_h u^{n+1} + \Delta_h u^n), \]  
\[ S = s(\tau \Delta_h) = (1 + \frac{\tau}{2} \Delta_h)(1 - \frac{\tau}{2} \Delta_h)^{-1} \]

The best results are for $L$-stable methods, i.e. 
\[ |s(z)| \leq 1 \text{ for } \Re z \leq 0 \text{ (A-stable)} \]  
\[ s(z) \rightarrow 0 \text{ as } z \rightarrow \infty \]
Examples: backward Euler, TGA (an improvement of CN); one SDIRK2 (Runge-Kutta); BDF2 (but multi-step) 
CN is A-stable, but not L-stable; $s(\infty) \neq 0$.  

**Main Result.** For an L-stable, single step method with operator norm on $L^\infty(\mathbb{R}_h^d)$,

$$\|S^n\| \leq C_0$$
$$\|D_hS^n\| \leq C_1(n\tau)^{-1/2}$$
$$\|D_h^2S^n\| \leq C_2(n\tau)^{-1}$$

for $n\tau \leq T$, with constants indep’t of $h$, $\tau$; similarly BDF2. For CN, more limited estimates: $\|S^n\| \leq C_0$ and

$$\|D_hRS^n\| \leq C_1(n\tau)^{-1/2}, \quad R = (1 - \frac{\tau}{2}\Delta_h)^{-1}$$


Related work: Aronson ('63), Widlund ('66) for $\tau = O(h^2)$; Ashyralyev & Sobolevskii; Thomée et al. for finite elements Michael Pruitt, Numer. Math. ('14) generalization to parabolic operators with variable coefficients
Proof of the Main Result

With \( S = \) time step operator, L-stable, single step, in max norm
\[
\| S^n \| \leq C_0 \\
\| D_h S^n \| \leq C_1 (n\tau)^{-1/2} \\
\| D_h^2 S^n \| \leq C_2 (n\tau)^{-1}
\]

Use the point of view of analytic semigroups, as in
Thomée, Galerkin FEM for Parabolic Problems
Ashyralyev & Sobolevskii, ...Parabolic Difference Eq’ns

Proof in three steps:
(1) For the semidiscrete equation \( u_t = \Delta_h u \), \( u = u(jh, t) \)
estimate \( u \) and its differences for complex \( t \),
\[
\| D_h^m e^{\Delta_h t} \| \leq C_m |t|^{-m/2} \text{ on } L^\infty(\mathbb{R}^d_h)
\]
(2) Estimate \( D_h^m (z - \Delta_h)^{-1} \) using (1).
(3) Write \( s^n(z) \) as a contour integral using \( (z - \Delta_h)^{-1} \);
estimate \( D_h^m s^n(z) \) on \( L^\infty(\mathbb{R}^d_h) \) for \( n \) large.
Step 1. Semidiscrete: \( u_t = \Delta_h u \), \( u(t) = e^{\Delta_h t} u^0 \)

Show \( \|D_h^m e^{\Delta_h t}\| \leq C_m |t|^{-m/2} \) on \( L^\infty(\mathbb{R}_h^d) \), for complex \( t \)

\( t \) in a sector, \( \{ t = t_1 + it_2 : t_1 > 0, |t_2| \leq M t_1 \} \)

Use \( g_t = \Delta_h g \), \( g(jh, 0) = \delta_{j0} \)

\( u(jh, t) = \sum_\ell g(jh - \ell h, t)u^0(\ell h) \)

Estimate \( D_h^m g \) in discrete \( L^1 \) using transform (F. John, '52)

\[
 f(jh) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\xi) e^{i j \xi} \, d\xi, \quad \hat{f}(\xi) = \sum_{j \in \mathbb{Z}} f(jh) e^{-i j \xi}
\]

in one dimension; \( \hat{g}(\xi, t) = \exp(-4t \sin^2(\xi/2)/h^2) \); e.g.,

\[
 |g(jh, t)| \leq \frac{C}{j^2} \int_{-\pi}^{\pi} |\hat{g}''(\xi, t)| \, d\xi \leq \frac{C'}{j^2} \frac{|t|^{1/2}}{h}
\]

\[
 (D_h^+ g)^\wedge(\xi) = (2i/h) e^{i\xi/2} \sin(\xi/2)\hat{g}(\xi)
\]
Step 2. Estimate the resolvent of $\Delta_h$:

$$(z - \Delta_h)^{-1} = \int_0^\infty e^{-zt} e^{\Delta_h t} \, dt$$

Moving the ray in the $t$-sector, we get

$$\| (z - \Delta_h)^{-1} \| \leq C_0 |z|^{-1},$$
$$\| D_h (z - \Delta_h)^{-1} \| \leq C_1 |z|^{-1/2},$$
$$\| D_h^2 (z - \Delta_h)^{-1} \| \leq C_2 (1 + | \log |z| | + | \log h |)$$

for $z$ outside a sector about $z < 0$

For periodic functions, mean value zero,

$$\| (\Delta_h)^{-1} \| \leq C_0, \quad \| D_h (\Delta_h)^{-1} \| \leq C_1, \quad \| D_h^2 (\Delta_h)^{-1} \| \leq C_2 (1 + | \log h |)$$
Step 3. Estimate \( S^n = s(\tau \Delta_h)^n \) and differences:

\[
D_h^\alpha s(\tau \Delta_h)^n = \frac{1}{2\pi i} \int_{\Gamma} s(z)^n D_h^\alpha (z - \tau \Delta_h)^{-1} dz
\]

use contour \( \Gamma \), radius \( O(1/n) \)
use resolvent estimates
from Step 2
use assumptions on \( s \):

\( s(z) \) analytic in a sector about \( z < 0 \),
\( s(z) = 1 + z + O(z^2) \) as \( z \to 0 \),
\( |s(z)| \leq (1 + c_1|z|)^{-1} \), \( z \) in sector about \( z < 0 \)
(from consistency and L-stability)

Discretizing the Navier-Stokes equations

For exact solutions with periodic boundary conditions

\[
    u_t + u \cdot \nabla u + \nabla p = \Delta u + F, \quad \nabla \cdot u = 0
\]

Use \( P \), the \( L^2 \)-projection on divergence-free vector fields

\[
    u_t + P(u \cdot \nabla u) = \Delta u + PF
\]

where

\[
    Pw = w - \nabla \Delta^{-1} \nabla \cdot w
\]

To discretize in space, use centered diff’s for \( \nabla \), e.g.

\[
    \frac{\partial u}{\partial x_1} \approx \frac{u_{i+1,j} - u_{i-1,j}}{2h}
\]

We use the usual discrete Laplacian and get an “approximate projection”.

(A staggered grid (MAC) would avoid this.)
Use centered differences for grad and div, $\nabla \approx \nabla_h$

Suppose we define $\Delta_w u = \nabla_h \cdot \nabla_h u$, then $\Delta_w$ is the “the wide Laplacian”, in 2D

$$\Delta_w u = \frac{u_{i+2,j} + u_{i-2,j} + u_{i,j+2} + u_{i,j-2}}{4h^2}$$

$P_0 = I - \nabla_h \Delta_w^{-1} \nabla_h \cdot$ is a projection,

i.e. $P_0^2 = P_0$, “the exact discrete projection”

but it is preferable to use the usual discrete Laplacian,

$$\Delta_h u = \frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}}{h^2}$$

Then we get an “approximate projection” $\tilde{P}$, $\tilde{P}^2 \neq \tilde{P}$.

\[ \tilde{P} = I - \nabla_h \Delta_h^{-1} \nabla_h \cdot = \text{“approx projection”} \]
\[ P_0 = I - \nabla_h \Delta_w^{-1} \nabla_h \cdot = \text{“exact discrete projection”} \]

With periodic boundary conditions,
\[ \Delta_h = 0 \text{ on constants, invertible on fcns of mean value zero} \]
\[ \Delta_w \text{ has a null space with dim } 2^d, = 0 \text{ on constants and } (-1)^i. \]
\[ \Delta_w \text{ is invertible on } \perp \text{ to its null space} \]

On \( L^2 \), \( \| \tilde{P} \| \leq 1 \), from checking eigenvalues (\( L^2 \) stability)
On \( L^\infty \), \( \| \tilde{P} \| \leq C \log h \), using the elliptic estimate.
Also on \( L^\infty \), \( \| P_0 \| \leq C \log h \). (Grid size \( 2h \))
\[ D_h^2 \Delta_w^{-1} \text{ is bounded by } C \log h \text{ for centered differences!} \]

Often with interfaces a MAC (staggered) grid is used
so that the approx proj’n is an exact proj’n.
Navier-Stokes with a moving interface

periodic bdry cond’ns on box interface $\Gamma$, depending on $t$
force $f$ on $\Gamma$ acts on fluid
force on $\Gamma$ amounts to jumps in $p$ and $\nabla u$ (here in 2D)

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p + \mu \Delta u + f \delta_{\Gamma}$$

$$\nabla \cdot u = 0$$

$$[p] = f \cdot n, \quad \left[ \frac{\partial p}{\partial n} \right] = \frac{\partial}{\partial s} (f \cdot \tan),$$

$$[u] = 0, \quad \mu \left[ \frac{\partial u}{\partial n} \right] = - (f \cdot \tan) \tan.$$
The Scheme for Navier-Stokes

time step $\tau = O(h)$, viscosity $= 1$

\[ u^{n+1} - u^n = -\tau (u \cdot \nabla u)^{n+1/2} - \tau \nabla p^{n+1/2} + \tau \Delta u^{n+1/2} + \tau C_1 + \tau C_7 \]

$C_1$ corrects $u_t^{n+1/2} \approx (u^{n+1} - u^n)/\tau$ if crossing $\Gamma$

\[ (u \cdot \nabla u)^{n+1/2} = \frac{3}{2} u^n \cdot \nabla h u^n - \frac{1}{2} u^{n-1} \cdot \nabla h u^{n-1} + C_2 \]

\[ \Delta u^{n+1/2} = \frac{1}{2} (\Delta_h u^{n+1} + \Delta_h u^n) + C_3 \]

$C_2$ and $C_3$ correct differences using jumps in $Du, D^2u$

\[ \Delta_h p^{n+1/2} = - \left( \nabla_h \cdot (u \cdot \nabla u)^{n+1/2} + C_4 \right) + C_5 - m \]

$C_5$ uses jumps in $p, \partial p/\partial n, u \cdot \nabla u$

$m$ is a constant so RHS has mean value zero

\[ \nabla p^{n+1/2} = \nabla_h p^{n+1/2} + C_6 \]
This scheme is similar to ones that have been developed:
Z. Li, M.-C. Lai, JCP 2001
L. Lee, R. LeVeque, SISC 2003
D. Le, B. Khoo, J. Peraire, JCP 2006
S. Xu, Z. J. Wang, JCP 2006

We assume periodic boundary conditions in the computational box. With a solid boundary, the bdry cond’n must be discretized. Discretizing at the boundary and at the interface should be separate issues?

Often a MAC grid is used; the analysis should apply. Xu & Wang use explicit time stepping, larger Re.
**Theorem.** Assume the exact solution is good for $0 \leq t \leq T$. Neglect errors in the location of $\Gamma$. Assume $f$ and $\nabla_{\tan} f$ are known to $O(h^2)$. Choose time step $\tau$ with $\tau / h$ fixed. Then for $t_n = n\tau \leq T$

$$\max_{j,n} \left| u^{\text{computed}}(x_j, t_n) - u^{\text{exact}}(x_j, t_n) \right| \leq K_T h^2 |\log h|^2$$

Similarly the pressure error is bounded by $h^2 |\log h|^3$ except for an indefinite constant.
Error Estimate for NSE

\[ u, \ p \] computed quantities; \( v, \ q \) exact

\[ v^{n+1} - v^n = -\tau (v \cdot \nabla v)^{n+1/2} - \tau \nabla q^{n+1/2} + \tau \Delta v^{n+1/2} + \tau C_1 + \tau C_7 - \tau \varepsilon^n \]

Here \( q^{n+1/2} \) solves a Poisson problem like \( p^{n+1/2}, \) \( v \) in place of \( u \)

For example, the error in \( \Delta v^{n+1/2} \) is

\[ O(h^2) \] at regular points, \( O(h) \) at irregular points

It has the form \( O(h^2) + DO(h^2) = (I + D)O(h^2) \) by the Lemma

We show the truncation error is

\[ \varepsilon^n = (I + D)O(h^2|\log h|), \quad \varepsilon^0 = O(h|\log h|) \]

Let \( w = u - v = \) velocity error; subtract equations, cancel \( C \)'s!

Set \( g^{n+1/2} \equiv (u \cdot \nabla_h u)^{n+1/2} - (v \cdot \nabla_h v)^{n+1/2}. \) Then

\[ \nabla_h p^{n+1/2} - \nabla_h q^{n+1/2} = -\nabla_h \Delta_h^{-1} \nabla_h \cdot g^{n+1/2} = -(I - \tilde{P})g^{n+1/2} \]

\[ w^{n+1} - w^n = -\tau \tilde{P}g^{n+1/2} + (\tau/2)(\Delta_h w^{n+1} + \Delta_h w^n) + \tau \varepsilon^n \]
Error Estimate for NSE, page 2 of 4

\( w = u - v = \) velocity error; \( \tilde{P} = \) approx proj’n, \( P_0 = \) exact proj’n

\[
\begin{align*}
    w^{n+1} - w^n & = -\tau \tilde{P}g^{n+1/2} + \left( \frac{\tau}{2} \right) (\Delta_h w^{n+1} + \Delta_h w^n) + \tau \varepsilon^n \\
    R & = (I - \frac{\tau}{2} \Delta_h)^{-1}, \quad S = (I + \frac{\tau}{2} \Delta_h)(I - \frac{\tau}{2} \Delta_h)^{-1} \\
    w^{n+1} & = S w^n - \tau R \tilde{P}g^{n+1/2} + \tau R \varepsilon^n
\end{align*}
\]

Since \( \| \tilde{P} \| \sim | \log h | \), a direct stability estimate doesn’t work?

Define \( y^n = P_0w^n \) and \( z^n = (I - P_0)w^n \), and estimate separately.

\( \| w^n \| \leq \| y^n \| + \| z^n \| \) but not the reverse!

\[
\begin{align*}
    y^{n+1} & = S y^n - \tau R P_0 g^{n+1/2} + \tau P_0 R \varepsilon^n \\
    z^{n+1} & = S z^n - \tau R A (I - P_0)g^{n+1/2} + \tau (I - P_0) R \varepsilon^n
\end{align*}
\]

where \( A = (\Delta_h - \Delta_w) \Delta_h^{-1} \) is bounded indep’t of \( h \)
(proof uses the Fourier multiplier)
\[ y^{n+1} = Sy^n - \tau RP_0 g^{n+1/2} + \tau P_0 R \varepsilon^n \]
\[ y^n = P_0 w^n; \quad g^{n+1/2} = (u \cdot \nabla_h u)^{n+1/2} - (v \cdot \nabla_h v)^{n+1/2} \]
We know \( g \) is bounded like \( \| D_h y \| + \| D_h z \| \).
We need to know the same for \( P_0 g = g - \nabla_h \Delta_w^{-1}(\nabla_h \cdot g) \).
How to avoid the log \( h \) from \( P_0 \)?
\[ g = v \cdot \nabla_h w + \cdots = v \cdot \nabla_h y + v \cdot \nabla_h z + \cdots \]
For exact equations, \( \nabla \cdot (v \cdot \nabla v) = \sum v_j, i v_i, j \) since \( \nabla \cdot v = 0 \).
Here \( y^n = P_0 w^n, \nabla_h \cdot y^n = 0, \) and the same works.
\[ z^n = (I - P_0) w^n \] is a discrete gradient, \( D_{hi} z_j = D_{hj} z_i \) and
\[ (I - P_0)(v \cdot \nabla_h z) = \nabla_h \Delta_w^{-1} \Delta_w (v_j z_j) + D_h B(z) = \nabla_h (v_j z_j) + D_h B(z) \]
All together \( P_0 g^{n+1/2} = \Phi^n_0 + D_h \Phi^n_1 + \) (same \( (n-1) \))
where \( \| \Phi^n_k \| \leq K(\| y^n \| + \| z^n \|), k = 0 \text{ or } 1 \)
It is important that \( v \) is continuous so that \( Dv \) is bounded!
Error Estimate for NSE, concluded

\[ y^{n+1} = Sy^n - \tau RP_0 g^{n+1/2} + \tau P_0 R\varepsilon^n \]

\[ y^{n+1} = -\tau \sum_{\ell=1}^{n} S^{n-\ell} RP_0 g^{\ell+1/2} + \tau \sum_{\ell=0}^{n} S^{n-\ell} RP_0 \varepsilon^{\ell} \]

We use \[ \|S^{n-\ell} RD_h\| \sim ((n - \ell)\tau))^{-1/2}, \quad P_0 g \sim Dy + Dz \]
Define \( \delta^n = \max (\|y^m\| + \|z^m\|) , \ m \leq n \)

\[ |\text{first sum}| \sim \sum_{\ell=1}^{n-1} ((n - \ell)\tau))^{-1/2} \delta^\ell \tau + \tau^{-1/2} \delta^n \tau \]

Similarly for the second sum, with \( \varepsilon^\ell \sim (I + D)O(h^2 |\log h|^2) \).
Add a similar estimate for \( z^n \) to get

\[ \delta^{n+1} \leq K_1 \sum_{\ell=1}^{n-1} ((n - \ell)\tau))^{-1/2} \delta^\ell \tau + K_1 \tau^{-1/2} \delta^n \tau + K_2 h^2 (\log h)^2 \]

and finally, with Hölder and Grönwall,

\[ \delta^{n+1} \leq Kh^2 (\log h)^2 \]


Michael Pruitt, Numerische Mathematik 128 (2014) discrete diffusion eq’ns, variable coeff’ts
Another project:


W. Ying and JTB, ”A fast accurate boundary integral method for potentials on closely packed cells,” CiCP 14 (2013)

S. Tlupova and JTB, ”Nearly singular integrals in 3D stokes flow,” CiCP 14 (2013)
