A new 3D immersed boundary method with application

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Talk Schedule

• 1) Background/motivation
• 2) Mathematical formulation
• 3) Numerical methods and algorithm
• 4) Verification/validation
• 5) Application
• 6) Summary
• 7) Future work
• **Immersed Boundary Methods**

• **Immersed Interface Methods**

• **Immersed Finite Element Methods**

• **Fictitious Domain Methods**

• **Arbitrary Lagrangian Eulerian Methods**

• **Phase Field Methods**

• ……..
The immersed boundary method (Charles Peskin 1972) is a practical and effective method for problems of this kind.

- The boundary can be active (*beating heart, swimming sperm*) or passive (*flag-in-wind*)
- The boundary can be neutrally buoyant (*swimming fish*) or can have higher or lower density than surrounding fluids (*aggregated RBCs in flowing blood*)
- May be a body (*eel, RBC*) or a surface (*flag, paper*)
- May be open (*flag*) or closed (*balloon*)
- The boundary may be modeled by a collection of discrete elastic springs/fibers or by continuum/solid mechanics
Different versions of the IB method (incomplete list)


**Vortex-method version** (McCracken & Peskin 1980)

**Volume-conserved version** (Peskin & Printz 1993, Rosar & Peskin 2001)

**Adaptive mesh version** (Roma, Peskin & Berger 1999, Griffith & Peskin 2007)

**Second-order version** (Lai & Peskin 2000, Griffith & Peskin 2005)

**Multigrid version** (Fogelson & Zhu, Zhu & Peskin 2002)


**Stochastic version** (Atzberger, Kramer & Peskin 2006)

**Kirchhoff rod version** (Lim, Ferent, Wang & Peskin 2008)

**Viscoelastic fluid version** (Chrispell, Cortez, Khismatullin, Fauci 2011, Chrispell, Fauci, Shelley 2013)


**Finite element version** (B Griffith & XY Luo 2014)

**Lattice Boltzmann version** (Zhu, He, et al. 2010)

Why another version of IB?
Motivation – FSI involving thin-walled-structures
Challenge of modeling thin-walled-structure by existing IB methods

- Elastic spring/fiber or 3D finite elements
- Lagrangian grid size for solid in the thickness direction is very small
- Accordingly the fluid grid has to be as small as the finest solid grid size to capture the thin yet finite thickness; may be too costly for computing in 3D
- A quick fix --- AMR
- Alternative approach: a collection of thin flat plates model the thin-walled structure
Mathematical formulation of the IB method for fluid-thin-walled-structure interaction

**FLUID**

\[ \nabla \cdot \mathbf{v} = 0 \]

\[ \rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mu \nabla^2 \mathbf{v} + \mathbf{f} \]

**STRUCTURE**

\[ \rho_S \frac{Dd}{Dt} = \nabla \cdot (\mathbf{S} \cdot \mathbf{F}^T) + \rho_S \mathbf{f}_S \]

S -- 2nd Piola-Kirchhoff stress tensor, F -- deformation gradient gradient tensor

\[ \mathbf{v}_f = \mathbf{V}_s \]

\[ \mathbf{f}_f = \mathbf{f}_s \]
FSI by the penalty approach (Goldstein, Handler, Sirovich 1993, Kim & Peskin 2007, Huang, Shin, Sung 2007, Huang, Chang, Sung 2011)

**FSI inner B.C.**
- Velocity: \( u_f = V_s \)
- Force: \( f_f = f_s \)

**virtual IB** \( X_{IB} \)
- real IB \( X \)
- compute interacting force by

\[
F_{IB}(s_1, s_2, t) = \alpha \int_0^t \left[ V_{IB}(s_1, s_2, t') - V_s(s_1, s_2, t') \right] dt' + \beta \left[ V_{IB}(s_1, s_2, t) - V_s(s_1, s_2, t) \right]
\]

\[
f(x, t) = \int_{\Gamma} F_{IB}(s_1, s_2, t) \delta(x - X_{IB}(s_1, s_2, t)) ds_1 ds_2
\]

\[
\rho \left( \frac{\partial v}{\partial t} + v \cdot \nabla v \right) = -\nabla p + \mu \nabla^2 v + f
\]

\[
\rho_s \frac{Dd}{Dt} = \nabla \cdot (S \cdot F^T) + \rho_s f_s
\]
existence and uniqueness of solutions of the mathematical formulation

$$\nabla \cdot \mathbf{v} = 0$$

$$\rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mu \nabla^2 \mathbf{v} + \mathbf{f}$$

$$\rho_s \frac{D\mathbf{d}}{Dt} = \nabla \cdot (\mathbf{S} \cdot \mathbf{F}^T) + \rho_s \mathbf{f}_s$$

$$\mathbf{F}_{IB}(s_1, s_2, t) = \alpha \int_0^t \left[ \mathbf{V}_{IB}(s_1, s_2, t') - \mathbf{V}_s(s_1, s_2, t') \right] dt' + \beta \left[ \mathbf{V}_{IB}(s_1, s_2, t) - \mathbf{V}_s(s_1, s_2, t) \right]$$

$$\mathbf{f}(\mathbf{x}, t) = \int_{\Gamma} \mathbf{F}_{IB}(s_1, s_2, t) \delta(\mathbf{x} - \mathbf{X}_{IB}(s_1, s_2, t)) ds_1 ds_2$$

$$\mathbf{V}_{IB}(s_1, s_2, t) = \int_{\Gamma} \mathbf{v}(\mathbf{x}, t) \delta(\mathbf{x} - \mathbf{X}_{IB}(s_1, s_2, t)) d\mathbf{x}$$

$$\frac{\partial \mathbf{X}_{IB}}{\partial t}(\mathbf{\alpha}, t + 1) = \mathbf{V}_{IB}(\mathbf{\alpha}, t + 1)$$
Hybrid numerical methods (FEM + LBM)

• Lattice Boltzmann (D3Q19) for the Navier Stokes

• Finite element for the solid equations (Co-rotational scheme)

• Finite difference for other eqns
finite element method for thin plate motion

\[ \rho_s \frac{Dd}{Dt} = \nabla \cdot (S \cdot F^T) + \rho_s f_s \]

large deformation
large displacement
large rotation

nonlinearities in
material & geometry

small deformation
(linear elasticity)

co-rotational scheme
for geometrical
nonlinearity

global

local
FEM for in-plane & out-of-plane motions

Stiff matrix/load vector are computed in local coordinates

**In-plane motion by MRT**

\[
\rho_s h \frac{\partial^2 u}{\partial t^2} = \frac{Eh}{1 - \nu^2} \left[ \nabla^2 u - \frac{1}{2} (1 + \nu) \left( \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x \partial y} \right) \right] + F_{ext}^x,
\]

\[
\rho_s h \frac{\partial^2 v}{\partial t^2} = \frac{Eh}{1 - \nu^2} \left[ \nabla^2 v - \frac{1}{2} (1 + \nu) \left( \frac{\partial^2 v}{\partial y^2} - \frac{\partial^2 u}{\partial x \partial y} \right) \right] + F_{ext}^y,
\]

**Out-of-plane motion by DKT**

\[
\rho_s h \frac{\partial^2 w}{\partial t^2} = -\frac{Eh^3}{12(1 - \nu^2)} \nabla^2 \nabla^2 w + F_{ext}^z,
\]

Stiff matrix & load vector are transformed to and assembled in global coordinates by change of coordinates via transformation matrix \( T^e \)

\[
K^e_T = T^e^T \left[ K^e_E + K^e_G \right] T^e
\]

\[
F^e_T^e = T^e F^e
\]
Newmark for time discretization

Assembly stiff matrix & load vector in global coordinate system

\[ \mathbf{K} = \sum_{e} K^e \quad \mathbf{F} = \sum_{e} F^e \]

semi-discretization by FEM

\[ \dddot{\mathbf{d}} + \mathbf{F}_i(\ddot{\mathbf{d}}) = \mathbf{F} \]

Newmark method for ODE system

Newton’s method for the nonlinear algebraic system

\[ (\mathbf{K} + \frac{4}{\Delta t} \mathbf{M}) \Delta \ddot{\mathbf{d}} = \text{RHS} \]

LDL for linear algebraic system of each Newton step

\[ \mathbf{A} = \text{LDL} \]
N-S eqns by lattice Boltzmann method

A) Why LB method for Navier-Stokes?
1) Lattice Boltzmann method solves a series of linear scalar differential equations
2) Relationship between pressure and density
3) Clear physical interpretation of the scheme
4) Natural for parallelization
5) Easier to model extra physics in a flow problem

B) A brief introduction to the LB method

C) Lattice-Boltzmann based IB method
Some References for the LBM

Books on the Lattice Boltzmann Method:


Widely cited paper on the LBM:


LBM is a fast growing area
I) Single Component LBM

Boltzmann Equation (1872), PDE for velocity distribution $f(x, \xi, t)$

$$\frac{\partial f(x, \xi, t)}{\partial t} + \xi \cdot \frac{\partial f(x, \xi, t)}{\partial x} + \frac{F(x, t)}{m} \cdot \frac{\partial f(x, \xi, t)}{\partial \xi} = Q(f, f)$$

BGK model, 1954

$$\frac{\partial f(x, \xi, t)}{\partial t} + \xi \cdot \frac{\partial f(x, \xi, t)}{\partial x} = -\frac{1}{\tau}(f(x, \xi, t) - f^0(x, \xi, t)),$$

$\xi$ can be discretized by $\{\xi_j, j = 0, 1, 2, \ldots n.\}$
Two widely used lattice Boltzmann models
(left: D2Q9  right: D3Q19)
Discrete lattice BGK model

\[ \frac{\partial f_j(x,t)}{\partial t} + \xi_j \cdot \frac{\partial f_j(x,t)}{\partial x} = -\frac{1}{\tau} (f_j(x,t) - f_j^0(x,t)) \]

Discretization in time

\[ \xi = \frac{dx}{dt} \]

\[ \frac{df(x,t)}{dt} = -\frac{1}{\tau} (f(x,t) - f^0(x,t)) \]

The lattice Boltzmann equation (LBE)

\[ f_j(x + \xi_j, t + 1) = f_j(x, t) - \frac{1}{\tau} (f_j(x, t) - f_j^0(x, t)) \]
\[
\frac{\partial f(x, \xi, t)}{\partial t} + \xi \cdot \frac{\partial f(x, \xi, t)}{\partial x} + \frac{F(x, t)}{m} \cdot \frac{\partial f(x, \xi, t)}{\partial \xi} = Q(f, f)
\]

Simplified Boltzmann equation by BGK

\[
\frac{\partial f(x, \xi, t)}{\partial t} + \xi \cdot \frac{\partial f(x, \xi, t)}{\partial x} = -\frac{1}{\tau}(f(x, \xi, t) - f^0(x, \xi, t))
\]

Discrete lattice BGK equation

\[
\frac{\partial f_j(x, t)}{\partial t} + \xi_j \cdot \frac{\partial f_j(x, t)}{\partial x} = -\frac{1}{\tau}(f_j(x, t) - f^0_j(x, t))
\]

The lattice Boltzmann equation (LBE)

\[
f_j(x + \xi_j, t + 1) = f_j(x, t) - \frac{1}{\tau}(f_j(x, t) - f^0_j(x, t))
\]
Physical interpretation of lattice Boltzmann method

\[ f_j(x + \xi_j, t + 1) = f_j(x, t) - \frac{1}{\tau}(f_j(x, t) - f_j^0(x, t)) \]

1) Collision

\[ f_j^*(x, t) = f_j(x, t) - \frac{1}{\tau}(f_j(x, t) - f_j^0(x, t)) \]

2) Streaming

\[ f_j(x + \xi_j, t + 1) = f_j^*(x, t) \]

3) Bounce-back BC
Literature on using LB method in the IB method


X Wang, C. Shu, J. Wu & LM Yang, Computer & Fluids, 100, pp. 165-175 (2014)
Some features of our new LB-IB method

1) Our new method is formulated for a thin-walled structure, and the motion of the structure is described by thin-plate eqns.

2) The fluid equations are solved by the D3Q19 LB model and structure equations are solved by an especial finite element method. Other equations in the formulation are solved by the finite difference methods.

3) The fluid-structure interaction forces are computed by the penalty approach.
Lattice Boltzmann (D3Q19) for N-S eqns

\[
g_j(x+\xi_j, t+1) = g_j(x, t) - \frac{1}{7} (g_j(x, t) - g_j^0(x, t)) + (1 - \frac{1}{2\tau}) \omega_j \left( \frac{\xi_j - v(x, t)}{c_s^2} + \frac{\xi_j \cdot v(x, t)}{c_s^4} \xi_j \right) \cdot f(x, t)
\]

\[
F_{IB}(s_1, s_2, t) = \alpha \int_0^t [V_{IB}(s_1, s_2, t') - V_s(s_1, s_2, t')] dt' + \beta [V_{IB}(s_1, s_2, t) - V_s(s_1, s_2, t)]
\]

\[
f(x, t) = \sum_\alpha F_{IB}(\alpha, t) \delta_h(x - X_{IB}(\alpha, t)) \Delta \alpha
\]

\[
\rho(x, t + 1) = \sum_j g_j(x, t + 1)
\]

\[
(\rho v)(x, t + 1) = \sum_j \xi_j g_j(x, t + 1) + \frac{f(x, t)}{2}
\]

\[
V_{IB}(\alpha, t + 1) = \sum_x v(x, t + 1) \delta_h(x - X_{IB}(\alpha, t)) \Delta x
\]

\[
\frac{X_{IB}(\alpha, t + 1) - X_{IB}(\alpha, t)}{\Delta t} = V_{IB}(\alpha, t + 1)
\]
Summary: algorithm (from $t$ to $t+1$)

\[ F_{IB}(s_1, s_2, t) = \alpha \int_0^t \left[ V_{IB}(s_1, s_2, t') - V_s(s_1, s_2, t') \right] dt' + \beta \left[ V_{IB}(s_1, s_2, t) - V_s(s_1, s_2, t) \right] \]

\[ f(x, t) = \sum_{\alpha} F_{IB}(\alpha, t) \delta_h(x - X_{IB}(\alpha, t)) \Delta \alpha \]

\[ g_j(x+\xi_j, t+1) = g_j(x, t) - \frac{1}{\tau}(g_j(x, t) - g_j^0(x, t)) + (1 - \frac{1}{2\tau})w_j \left( \frac{\xi_j - v(x, t)}{c_s^2} + \frac{\xi_j \cdot v(x, t)}{c_s^4} \right) \xi_j \cdot f(x, t) \]

\[ \rho(x, t+1) = \sum_j g_j(x, t+1) \]

\[ (\rho v)(x, t+1) = \sum_j \xi_j g_j(x, t+1) + \frac{f(x, t)}{2} \]

\[ V_{IB}(\alpha, t+1) = \sum_x v(x, t+1) \delta_h(x - X_{IB}(\alpha, t)) \Delta x \]

\[ \frac{X_{IB}(\alpha, t+1) - X_{IB}(\alpha, t)}{\Delta t} = V_{IB}(\alpha, t+1) \]

\[ \rho_s h \frac{\partial^2 u}{\partial t^2} = \frac{Eh}{1-\nu^2} \left[ \nabla^2 u - \frac{1}{2}(1+\nu) \left( \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 v}{\partial x \partial y} \right) \right] + F_{ext}^x \]

\[ \rho_s h \frac{\partial^2 v}{\partial t^2} = \frac{Eh}{1-\nu^2} \left[ \nabla^2 v - \frac{1}{2}(1+\nu) \left( \frac{\partial^2 v}{\partial y^2} - \frac{\partial^2 u}{\partial x \partial y} \right) \right] + F_{ext}^y, \]

\[ \rho_s h \frac{\partial^2 w}{\partial t^2} = -\frac{Eh^3}{12(1-\nu^2)} \nabla^2 \nabla^2 w + F_{ext}^z, \]

X($\alpha$, $t+1$) & Vs($\alpha$, $t+1$)

next time step $t+2$
Verification/Validation in lieu of numerical analysis

\[ \text{Re} = \frac{\rho_f U D}{\mu} = 100 \]

\[ M = \frac{h \rho_s}{L \rho_f} = 0.57 \]

\[ E^* = \frac{E}{(\rho_f U^2)} = 1.4 \times 10^3 \]

\[ \nu = 0.4 \]

Comparison with Turek & Hron (2006)
Verification/Validation (II)

(a) Cd

- Present Results
- Shenoy & Kleinsteuber (2008)
- Ross & Willmarth (1971)

(b) Rear vortex length

Flow past rigid plate

Re = 100, M = 1.0, K = 0.0001, S = 1000

Flow past a flag

Re = 100, M = 1.0, K = 0.0001, S = 1000

rear edge bottom point

rear edge middle point
Application

FIG. 2 (color online). Draping modes of a circular plate of radius $R = 0.05$ m. When the flow velocity is gradually increased, different draping patterns become visible starting from “mode C,” a cylindrical mode (a), to “mode 2F,” a twofold conical mode (b), “mode 3F,” a threefold mode (c), and eventually “mode 3F*,” a bent threefold mode (d).

$$Re - 10^5, \quad M - 10^{-2}, \quad S - 10^5$$

FIG. 1. (a) Schematic diagram of the flow over a flexible circular plate and (b) two coordinate systems: a moving curvilinear coordinate system \((s_1, s_2)\) for the plate and a fixed Cartesian coordinate system \((x, y, z)\) for the flow.

Spatial: \(h = 0.025D\)
Temporal: \(\Delta t = 0.0025\)

\[
Re = \frac{\rho_f U D}{\mu} \quad K = \frac{EI}{\rho_f U^2 D^3}
\]

\[
M = \frac{\rho_s h}{\rho_f D} \quad S = \frac{E h}{\rho_f U^2 D}
\]

\[
d = D_c / D
\]

\(Re = 100 \text{ & } 200\)
\(M = 0.1 \text{ & } 2.0, \quad d = 0.1 \text{ - } 0.5\)

\(K = 10^{-5} \text{ - } 10^{-1}, \quad S = 10 \text{ - } 1000\)

\(Re > 10^5, \quad M > 10^{-2}, \quad S > 10^5\)
Deformation modes

Folding number increases with the decreases of $K_b$

$Re=100$, $M=0.1$, $d=0.1$, $S=100$
Deformation mode on $S$-$K$ plane

$S$ increases, $K$ decreases $\Rightarrow n$ increases

$M=0.1, \ d=0.1$
2 typical plate shapes and streamlines
2 typical plate shapes and vorticity contours
Vortex structure in the wake

Re=100, steady for all modes
Re=200, steady for 3F ~ 6F, unsteady for 2F and 0F

M=0.1, d=0.1, S=100
Cd of the deformable plate

\[ M = 0.1, \quad d = 0.1, \quad S = 100 \]

Cd decreases with the decrease of K or increase of flexibility
Strain energy of the deformable plate

\( M = 0.1, \ d = 0.1, \ S = 100 \)

\( K_b \) dominates \( K_p \)
Influence of fixed region size on deformation modes

\[ Re = 100, \ M = 0.1, \ S = 100, \ K = 10^{-4} \]

**n increases with d**

\[ n = C (1 - d)^{-3/4} \quad C = 5.02 \]

\[ C = \frac{1}{(2\sqrt{\pi})K^{-1/4}Re^{-1/8}S^{1/4}} \]

Computed n is consistent with theoretical n

Cerda & Mahadevan 2003, PRL 90, 074302
Influence of M on deformation mode

\( M=0.1 \) (water tunnel)

\( M=2.0 \) (wind tunnel)

\( Re=100, \ d=0.1, \ S=100, \ K=10^{-4} \)

M only prolongs the evolution process to the final stage, no effects on the final mode.
Developing process of draping pattern

$Re=100, \quad M=2.0, \quad S=100, \quad K=0.0001$
Effect of Reynolds number on draping pattern

\[ M=2.0, S=1000, K=0.0001 \]

\[ Re=100 \quad 200 \quad 500 \]
Summary

1) A new 3D IB method for fluid-thin-walled-structure-interaction is introduced where immersed structure is modeled by thin plate equations and solved by the co-rotational finite element scheme and the fluid equations are solved by the lattice Boltzmann method. The new method is verified and validated via comparisons with existing works.

2) The new method is applied to a 3D viscous flow past a deformable thin circular plate. Our main findings are: a) There exist six different plate deformation patterns; b) bending energy $K_b$ dominates the plate strain energy; c) mass ratio $M$ does not alter the deformation pattern of the plate; d) drag coefficient $C_d$ decreases with the increase of structure flexibility; e) folding number increases with the size of the fixed region.
Ongoing and Future Work

1) Spectral/hp elements to model the thin-walled structures for extension to large deformation, displacement, and rotation, i.e. consider both material and geometry nonlinearities. (with S Dong and F Song)

2) Applications to Blood flows
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THE END

Thank you for your attention!