An immersed boundary method for simulating vesicle dynamics in three dimensions

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Outline

1. Introduction
2. Mathematical Model
3. Numerical Scheme
4. Numerical Experiments
5. Future Works
What are Vesicles?

Vesicles are cellular organelles that are composed of a lipid bilayer.

Figure: Diagram of lipid vesicles showing a solution of molecules (green dots) trapped in the vesicle interior. (Taken from Wikipedia.org)
Vesicle and Red Blood Cells (RBCs) both share similar mechanical behaviors.

**Figure:** Red Blood Cells (Taken from Google Search)
Governing equations

\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \frac{1}{Re} \Delta \mathbf{u} + \mathbf{f},
\]

(1)

\[
\nabla \cdot \mathbf{u} = 0,
\]

(2)

\[
f(x) = \int_{\Gamma} \left( \mathbf{F}_\sigma + \frac{1}{ReCa} \mathbf{F}_b \right) \delta(x - X(r, s, t)) dA,
\]

(3)

\[
\mathbf{F}_\sigma = \nabla_s \sigma - 2H\sigma \mathbf{n}, \quad \mathbf{F}_b = \frac{1}{2} (\Delta_s H + 2H(H^2 - K)) \mathbf{n},
\]

(4)

\[
\frac{\partial \mathbf{X}}{\partial t}(r, s, t) = \mathbf{U}(r, s, t) = \int_{\Omega} \mathbf{u}(x, t) \delta(x - X(r, s, t)) d\mathbf{x},
\]

(5)

\[
\nabla_s \cdot \mathbf{U} = 0 \quad \text{on } \Gamma.
\]

(6)

where \(Re = \rho R_0^2 / \mu t_c\) (the Reynolds number), \(Ca = \mu R_0^3 / c_b t_c\) (the capillary number), and \(R_0 = \sqrt{A/4\pi} = (3V/4\pi)^{1/3}\) (effective radius).

\[
\nu = \frac{3V}{4\pi(A/4\pi)^{3/2}}
\]

(7)

The reduced volume \(\nu\) represents the volume ratio between the vesicle and the sphere with the same surface area. \((\nu = 1\) for sphere)
Classical differential geometry

For the vesicle boundary $X$, define the first fundamental coefficients as

$$E = X_r \cdot X_r, \quad F = X_r \cdot X_s, \quad \text{and} \quad G = X_s \cdot X_s,$$

then

$$\nabla_s \sigma = \frac{GX_r - FX_s}{EG - F^2} \sigma_r + \frac{EX_s - FX_r}{EG - F^2} \sigma_s$$

$$\nabla_s \cdot U = \frac{GX_r - FX_s}{EG - F^2} \cdot U_r + \frac{EX_s - FX_r}{EG - F^2} \cdot U_s$$

Using $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$, we obtain two useful relations

$$X_s \times n = \frac{GX_r - FX_s}{|X_r \times X_s|}, \quad n \times X_r = \frac{EX_s - FX_r}{|X_r \times X_s|}$$

which give

$$\nabla_s \sigma = \frac{(X_s \times n) \sigma_r + (n \times X_r) \sigma_s}{|X_r \times X_s|} \quad \text{and}$$

$$X_s \times n_r + n_s \times X_r = -2Hn|X_r \times X_s|.$$
Nearly incompressible surface

To avoid solving the extra unknown tension $\sigma(r, s, t)$, we propose an alternative way to *approximate* the zero surface divergence. We use

$$\frac{\partial}{\partial t} |\mathbf{X}_r \times \mathbf{X}_s| = (\nabla_s \cdot \mathbf{U}) |\mathbf{X}_r \times \mathbf{X}_s|$$

from which each surface dilating factor should maintain the initial profile during time integration whenever $\nabla_s \cdot \mathbf{U} = 0$. Therefore, we introduce a spring-like elastic tension as

$$\sigma = \sigma_0 \left( |\mathbf{X}_r \times \mathbf{X}_s| - |\mathbf{X}_r^0 \times \mathbf{X}_s^0| \right)$$

where $\sigma_0 \gg 1$ and $|\mathbf{X}_r^0 \times \mathbf{X}_s^0|$ is the initial surface dilating factor. We also define the modified elastic energy by

$$E_\sigma(\mathbf{X}) = \frac{\sigma_0}{2} \iint (|\mathbf{X}_r \times \mathbf{X}_s| - |\mathbf{X}_r^0 \times \mathbf{X}_s^0|)^2 \, drds.$$
Derivation of the rate of change of surface dilation factor

$$\frac{\partial}{\partial t} |X_r \times X_s| = \frac{X_r \times X_s}{|X_r \times X_s|} \cdot (X_{rt} \times X_s + X_r \times X_{st})$$

$$= n \cdot (X_{rt} \times X_s) + n \cdot (X_r \times X_{st}) \quad \left(\text{since } n = \frac{X_r \times X_s}{|X_r \times X_s|}\right)$$

$$= (X_s \times n) \cdot X_{rt} + (n \times X_r) \cdot X_{st} \quad \left(\text{using } (a \times b) \cdot c = (b \times c) \cdot a\right)$$

$$= (X_s \times n) \cdot U_r + (n \times X_r) \cdot U_s \quad \left(\text{since } X_t = U\right)$$

$$= \frac{G X_r - F X_s}{|X_r \times X_s|} \cdot U_r + \frac{E X_s - F X_r}{|X_r \times X_s|} \cdot U_s \quad \left(\text{using the two relations}\right)$$

$$= (\nabla_s \cdot U)|X_r \times X_s| \quad \left(\text{by the definition of } \nabla_s \cdot U\right)$$
Derivation of modified elastic force by variational derivative

\[
\frac{d}{d\varepsilon} E_\sigma(X + \varepsilon Y) \bigg|_{\varepsilon=0} = \int\int \sigma_0 (|X_r \times X_s| - |X^0_r \times X^0_s|) \frac{X_r \times X_s}{|X_r \times X_s|} \cdot (Y_r \times X_s + X_r \times Y_s) \, drds
\]

\[
= \int\int \sigma n \cdot (Y_r \times X_s + X_r \times Y_s) \, drds \quad \text{(by } n = \frac{X_r \times X_s}{|X_r \times X_s|})
\]

\[
= \int\int \sigma (X_s \times n) \cdot Y_r + \sigma (n \times X_r) \cdot Y_s \, drds \quad \text{(by the scalar triple product formula)}
\]

\[
= -\int\int (\sigma X_s \times n)_r \cdot Y + (\sigma n \times X_r)_s \cdot Y \, drds \quad \text{(by integration by parts)}
\]

\[
= -\int\int [\sigma_r X_s \times n + \sigma_s n \times X_r + \sigma (X_s \times n)_r + \sigma (n \times X_r)_s] \cdot Y \, drds
\]

\[
= -\int\int (\sigma_r X_s \times n + \sigma_s n \times X_r + \sigma X_s \times n_r + \sigma n_s \times X_r) \cdot Y \, drds
\]

\[
= -\int\int (\nabla_s \sigma - 2\sigma Hn) \cdot Y \, |X_r \times X_s| \, drds
\]

\[
= -\int\int (\nabla_s \sigma - 2\sigma Hn) \cdot Y \, dA \quad \text{(since } dA = |X_r \times X_s| \, drds)
\]

\[
= -\int_\Gamma F_\sigma \cdot Y \, dA
\]
The unit outward normal vector of the $\ell$-th triangle is
\[
n_\ell = \frac{(\mathbf{X}_\ell^2 - \mathbf{X}_\ell^1) \times (\mathbf{X}_\ell^3 - \mathbf{X}_\ell^1)}{|(\mathbf{X}_\ell^2 - \mathbf{X}_\ell^1) \times (\mathbf{X}_\ell^3 - \mathbf{X}_\ell^1)|}.
\]

The area of the $\ell$-th triangle is
\[
dA_\ell = \frac{|(\mathbf{X}_\ell^2 - \mathbf{X}_\ell^1) \times (\mathbf{X}_\ell^3 - \mathbf{X}_\ell^1)|}{2}.
\]
**Numerical algorithm**

1. Compute the vesicle boundary forces.

We find the tension $\mathbf{F}_\sigma(X_k) = \nabla_s \sigma(X_k) - \sigma(X_k) \mathbf{H}(X_k)$ using

$$\sigma_\ell = \tilde{\sigma}_0 \left( dA_\ell - dA^0_\ell \right), \text{ where formally } \tilde{\sigma}_0 = \sigma_0 / (drds),$$

$$\sigma(X_k) = \sum_{\ell \in T(X_k)} \sigma_\ell / 3,$$

$$\nabla_s \sigma_\ell = \frac{(X^3_\ell - X^1_\ell) \times n_\ell}{2 \, dA_\ell} \left( \sigma^2_\ell - \sigma^1_\ell \right) + \frac{n_\ell \times (X^2_\ell - X^1_\ell)}{2 \, dA_\ell} \left( \sigma^3_\ell - \sigma^1_\ell \right),$$

$$\nabla_s \sigma(X_k) = \sum_{\ell \in T(X_k)} \omega_\ell \nabla_s \sigma_\ell, \text{ where } \omega_\ell = \frac{dA_\ell / 3}{dA(X_k)},$$

$$\mathbf{H}(X_k) = \frac{1}{2 \, dA(X_k)} \sum_{\ell \in T(X_k)} n_\ell \times (X^3_\ell - X^2_\ell), \text{ where } \mathbf{H} = 2Hn.$$

For a smooth surface patch $S$, the last formula is a discrete version of

$$\int_S 2Hn \, dA = \oint_{\partial S} n \times d\mathbf{X},$$

and equivalent to the cotangent formula.
From the discrete version of bending energy

\[
E_b[\mathbf{X}] = \frac{c_b}{8} \sum_{k=1}^{N_v} |\mathbf{H}(\mathbf{X}_k)|^2 \, dA(\mathbf{X}_k),
\]

we obtain the bending force \( \mathbf{F}_b(\mathbf{X}_k)dA(\mathbf{X}_k) = \)

\[
\frac{c_b}{8} \sum_{\ell \in \mathcal{T}(\mathbf{X}_k)} \left( (H_\ell - n_\ell \cdot C_\ell) \left( \frac{1}{2} n_\ell \times E^k_\ell \right) + \frac{1}{2} C_\ell \times E^k_\ell + n_\ell \times h^k_\ell \right),
\]

where

\[
H_\ell = \frac{1}{3} \sum_{p \in V(\ell)} |\mathbf{H}(\mathbf{X}_p)|^2, \quad C_\ell = \frac{1}{dA_\ell} \sum_{p \in V(\ell)} \mathbf{E}^p_\ell \times \mathbf{H}(\mathbf{X}_p),
\]

\[
E^k_\ell = \mathbf{X}^3_\ell - \mathbf{X}^2_\ell, \quad h^k_\ell = \mathbf{H}(\mathbf{X}^3_\ell) - \mathbf{H}(\mathbf{X}^2_\ell).
\]

Thus, the boundary force becomes

\[
\mathbf{F}(\mathbf{X}^n_k)dA(\mathbf{X}^n_k) = \mathbf{F}_\sigma(\mathbf{X}^n_k)dA(\mathbf{X}^n_k) + \mathbf{F}_b(\mathbf{X}^n_k)dA(\mathbf{X}^n_k).
\]
2. Solve the Navier-Stokes

\[\rho \left( \frac{3u^* - 4u^n + u^{n-1}}{2\Delta t} + 2 (u^n \cdot \nabla_h) u^n - (u^{n-1} \cdot \nabla_h) u^{n-1} \right) \]

\[= -\nabla_h p^n + \mu \Delta_h u^* + \sum_{k=1}^{N_v} F(X^n_k) dA(X^n_k) \delta_h(x - X^n_k), \]

\[\Delta_h p^* = \frac{3\rho}{2\Delta t} \nabla_h \cdot u^*, \quad \frac{\partial p^*}{\partial n} = 0 \text{ on } \partial \Omega, \quad u^* = u^{n+1} \text{ on } \partial \Omega, \]

\[u^{n+1} = u^* - \frac{2\Delta t}{3\rho} \nabla_h p^*, \quad \nabla_h p^{n+1} = \nabla_h p^* + \nabla_h p^n - \frac{2\mu\Delta t}{3\rho} \Delta_h(\nabla_h p^*). \]

3. Update the new position

\[X_{k}^{n+1} = X_{k}^{n} + \Delta t \sum_{x} u^{n+1}(x) \delta_h(x - X_{k}^{n}) h^3 \]
Equivalence of the discrete mean curvature vector formula and the cotangent formula

We begin with the cotangent formula

$$H(X)dA(X) = \frac{1}{2} \sum_{j=1}^{N} (\cot \alpha_j + \cot \beta_j) (X - X_j),$$

where $X_j$ are the vertices within 1-ring of the vertex $X$, and the angles $\alpha_j$ and $\beta_j$ are the corresponding angles.
The righthand side becomes

\[
\frac{1}{2} \sum_{j=1}^{N} \left( \cot \alpha_j + \cot \beta_j \right) (X - X_j)
\]

\[
= -\frac{1}{2} \sum_{j=1}^{N} \cot \alpha_j (X_j - X) + \cot \beta_j (X_j - X)
\]

\[
= -\frac{1}{2} \sum_{j=1}^{N} \cot \alpha_j (X_j - X) + \cot \beta_{j-1} (X_{j-1} - X) \quad \text{(using the periodicity of index)}
\]

\[
= -\frac{1}{2} \sum_{j=1}^{N} \cot \alpha_j (X_3^\ell - X_1^\ell) + \cot \beta_{j-1} (X_2^\ell - X_1^\ell) \quad \text{(denoting the vertices in the } \ell\text{-th triangle)}
\]

\[
= -\frac{1}{2} \sum_{j=1}^{N} \cot \alpha_j X_{31} + \cot \beta_{j-1} X_{21}, \quad \text{(where } X_{ij} = X_i^\ell - X_j^\ell)\text{.}
\]

(8)

Using the identity \( \cot \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|} = \frac{\mathbf{a} \cdot \mathbf{b}}{2 \, \text{d}A} \), where \( \theta \) is the angle between the two vectors \( \mathbf{a} \) and \( \mathbf{b} \), for the \( \ell \)-th triangle with the vertices \( X_1^\ell, X_2^\ell, X_3^\ell \) and its area \( \text{d}A_\ell \), we obtain

\[
\cot \alpha_j = \frac{X_{12} \cdot X_{32}}{2 \, \text{d}A_\ell}, \quad \cot \beta_{j-1} = \frac{X_{13} \cdot X_{23}}{2 \, \text{d}A_\ell}.
\]
Substituting these cotangents into Eq. (8) and summing all the triangles $T(X)$ within 1-ring around the vertex $X$, we have

$$\frac{1}{2} \sum_{\ell \in T(X)} \frac{(X_{21} \cdot X_{32})X_{31} - (X_{31} \cdot X_{32})X_{21}}{2 \, dA_{\ell}},$$

which, using the vector triple product $(a \times b) \times c = (a \cdot c)b - (b \cdot c)a$, becomes

$$\frac{1}{2} \sum_{\ell \in T(X)} \frac{(X_{21} \times X_{31}) \times X_{32}}{2 \, dA_{\ell}} = \frac{1}{2} \sum_{\ell \in T(X)} \frac{(X_{21} \times X_{31}) \times X_{32}}{|X_{21} \times X_{31}|} \quad \text{and}
\quad \frac{1}{2} \sum_{\ell \in T(X)} n_{\ell} \times X_{32} = \frac{1}{2} \sum_{\ell \in T(X)} n_{\ell} \times (X_{3}^{\ell} - X_{2}^{\ell}).$$
Remesh of triangular surface

Figure: Re-meshing triangulation by edge addition (top) and deletion (bottom).
Mapping of local area during simulation

\[ \sigma_\ell = \tilde{\sigma}_0 (dA^t_\ell - dA^0_\ell) \]

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<tr>
<td>2</td>
<td>(1, 4, 5)</td>
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<td>3</td>
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**Figure:** Mapping of triangular area.
Numerical results

Numerical issues

- Accuracy check for mean curvature and bending force
- Study on different stiffness parameter $\tilde{\sigma}_0$
- Convergence of vesicle configuration and fluid velocity
- Numerical experiments
  - A suspended vesicle in quiescent flow
  - A vesicle in shear flow
  - A vesicle under the gravity

Numerical parameters

Unless otherwise stated, we use

- $Re = 1$, $Ca = 50$
- $\sigma_0 = 6 \times 10^5$, grid width $h = 6/128$
- The initial number of triangles is either 81920 or 327680
Accuracy check for mean curvature and bending force

Using spherical parametric coordinates $(\theta, \phi) \in [0, 2\pi] \times [0, \pi]$,

- Unit sphere: $\mathbf{X}(\theta, \phi) = (\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi)$,
- Ellipsoid: $\mathbf{X}(\theta, \phi) = (0.5 \cos \theta \sin \phi, 0.5 \sin \theta \sin \phi, \cos \phi)$,
- Biconcave surface: $\mathbf{X}(\theta, \phi) = (\cos \theta \sin \phi, \sin \theta \sin \phi, (0.1242 + 0.8012 \sin^2 \phi - 0.4492 \sin^4 \phi) \cos \phi)$.

Figure: Left: A triangulated biconcave surface and its cross-sectional view. Right: The comparison of mean curvature between numerical values (symbols) and exact values (solid lines) for three different surfaces. Here, the number of vertices is $N_v = 2562$ corresponding to 5120 triangles used.
The \( L^2 \) error of a function \( \psi \) is calculated as
\[
\sqrt{\sum_{k=1}^{N_v} \left| \psi^e(X_k) - \psi(X_k) \right|^2 dA(X_k)},
\]
where \( \psi^e(X_k) \) is the exact value and \( \psi(X_k) \) is the computed value.
Study on different stiffness parameter $\tilde{\sigma}_0$

In this test, we study how the stiffness number $\tilde{\sigma}_0$ affects the conservation of surface area and vesicle volume, and the total surface energy.

- Put an initially oblate vesicle $\mathbf{X}(\theta, \phi) = (\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi/3.5)$ in quiescent flow.
- Computational domain $\Omega = [-2, 2]^3$ with grid size $128^3$.
- Choose $\tilde{\sigma}_0 = 6 \times 10^4, 6 \times 10^5, 6 \times 10^6$.
- The meshwidth $h = 1/32$ and the time step size $\Delta t = h/16$.
- The number of triangles used in the initial vesicle surface is 81920 with 40962 number of vertices.
Figure: The comparison for three different stiffness parameters: \( \tilde{\sigma}_0 = 6 \times 10^4 (\bigtriangleup) \), \( 6 \times 10^5 (\square) \), and \( 6 \times 10^6 (\bigcirc) \). (a) the maximum relative error of the local surface area; (b) the relative error of the global surface area; (c) the relative error of the global volume; (d) the total energy.
**Convergence of vesicle configuration and fluid velocity**

- Relax the same oblate vesicle with $\nu = 0.643$ in a cube $[-2, 2]^3$
- The grid size $N = 32, 64, 128, 256$
- $\tilde{\sigma}_0 = (N/32)^2 10^4$ and $\Delta t = h/8$
- $\text{ratio} = \log_2(\|u_N - u_{2N}\|_\infty / \|u_{2N} - u_{4N}\|_\infty)$
- When $N = 32$, we use 5120 number of triangles which corresponds to 2562 number of vertices.

**Figure**: The ratios of convergence for the fluid velocity $(u, v, w)$ and the vesicle configuration $X$. Left: $N = 32$; Right: $N = 64$
Numerical experiments

A suspended vesicle in quiescent flow

- **Oblate surface:** \( \mathbf{X}(\theta, \phi) = (\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi / 3.5) \)
- **Prolate surface:** \( \mathbf{X}(\theta, \phi) = (0.2 \cos \theta \sin \phi, 0.2 \sin \theta \sin \phi, \cos \phi) \)
- **Oscillatory surface:**
  \[
  \mathbf{X}(\theta, \phi) = \left( \frac{3}{20} r(\phi) \cos \theta \sin \phi, \frac{3}{20} r(\phi) \sin \theta \sin \phi, \frac{3}{25} r(\phi) \cos \phi \right),
  \]  
  where 
  \[
  r(\phi) = \sqrt{\cos^2 \phi + 9 \sin^2 \phi + \cos^2(4\phi)}.  
  \]
- **\( \Omega = [-2, 2]^3 \),** \( c_b = 0.02 \), \( \tilde{\sigma}_0 = 6 \times 10^5 \), and \( \Delta t = h/16 \)
Figure: Suspended vesicles in quiescent flow. **Oblate vesicle** (two upper-left panels); **Prolate vesicle** (two upper-right panels); **Oscillatory vesicle** (lower panels)
A vesicle in shear flow

- Put a prolate vesicle in a simple shear flow $u = (z, 0, 0)$
- The non-dimensional shear rate $\chi = \mu R_0^3 / c_b$
- $\Omega = [-3, 3]^3$ and a small Reynolds number $Re = 10^{-3} \ll 1$ (Stokes flow regime)
- The reduced volume $\nu = 0.8, 0.85, 0.9, 0.95, \text{and } 0.975$ by fixing the effective radius $R_0 = 1$
Figure: The simulation setup of a vesicle motion in a shear flow (top) and the tank-treading motion of a vesicle with $\nu = 0.8$ in a shear flow with $\chi = 100$ (bottom).
Figure: The plot of the inclination angle (left) and the scaled mean angular velocity (right) as functions of reduced volume $\nu$ for different dimensionless shear rate $\chi$.

The frequency $\omega$ can be computed using $\omega = \frac{1}{N_v} \sum_{i=1}^{N_v} \frac{|\mathbf{r}_i\times\mathbf{v}_i|}{|\mathbf{r}_i|^2}$, where $\mathbf{r}_i$ and $\mathbf{v}_i$ are the position and velocity of the vertices projected on the $xz$-plane, respectively.
**Figure:** Tumbling motion of a vesicle with $\nu = 0.8$ and $\lambda = 40$ in a shear flow with $\chi = 100$. The computed time is up to $t = 24$.

To consider viscosity contrast between the interior and exterior of vesicle, instead of

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \frac{1}{Re} \Delta \mathbf{u} + \mathbf{f},$$  \hspace{1cm} (9)

we use

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \frac{1}{Re} \nabla \cdot [\mu(x)(\nabla \mathbf{u} + \nabla \mathbf{u}^T)] + \mathbf{f},$$ \hspace{1cm} (10)

where $\mu(x)$ is the dimensionless viscosity contrast.
Figure: The volume relative errors (left) and the surface area relative errors (right) for the cases of with or without the penalty volume conservation term Eq. (11). Here, \( \nu = 0.8 \) and \( \chi = 100 \). The viscosity contrast \( \lambda = 1 \) and \( \lambda = 40 \), respectively.

To conserve the vesicle volume, we add

\[
\mathbf{F}_v \, dA(X) = -C_v \left( \frac{V^t - V^0}{V^0} \right) \mathbf{n} \, dA(X),
\]

where \( C_v \) is a sufficiently large constant called a penalty parameter, \( V^t \) is the global volume of vesicle at time \( t \), and \( V^0 \) is the global volume of initially given vesicle.
A vesicle under the gravity

- **Prolate:** $\mathbf{X}(\theta, \phi) = (0.5 \cos \theta \sin \phi, 0.5 \sin \theta \sin \phi, \cos \phi)$
- **Oblate:** $\mathbf{X}(\theta, \phi) = (0.75 \cos \theta \sin \phi, 0.75 \sin \theta \sin \phi, 0.375 \cos \phi)$
- The prolate vesicle is placed at different tilted angles $\eta = 0, \pi/4$ and $\pi/2$ initially to see how the initial orientation affects the equilibrium shape
- To model this problem, simply add an interfacial force $F_g dA(\mathbf{X}) = (\rho^i - \rho^e)(\mathbf{g} \cdot \mathbf{X}) dA(\mathbf{X}) \mathbf{n}$ where $\rho^i$ and $\rho^e$ are interior and exterior fluid densities, respectively, and $\mathbf{g} = (0, 0, -1)$
Figure: The prolate vesicle with three different tilted angles $\eta = 0, \pi/4, \pi/2$ (first to third column) and oblate (fourth column) vesicle under the gravity. All the snapshots are taken at the same time.
Future Works

- High-order scheme to compute the curvatures and the Laplace-Beltrami operator
- Vesicle dynamics in extensional flow or in Poiseuille flow
- Multi-vesicle problems to mimic the behaviors of RBCs in capillary
- Viscosity contrast effects on vesicle dynamics
- Reynolds number effects on vesicle dynamics
Thank you for your attention!