Thoughts on indicators and density notions

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IMS-JSPS Joint Workshop on Mathematical Logic and the Foundations of Mathematics
January 16, 2016
Introducing...
- basic ideas of indicators and their (slight) generalization,
- several consequences of the indicator arguments,
- some conservation results of combinatorial principles.

“Indicators are useful to analyze the first-order part of combinatorial statements in second-order arithmetic.”
Nonstandard models of arithmetic

In this talk we will mainly use the base system $EFA = IΔ_0 + \exp$ or $\text{RCA}_0^*$, which consists of $IΔ_0^0 + \exp$ plus $Δ_1^0$-comprehension, and models we will consider will be countable nonstandard. Let $M \models EFA$.

- $I \subseteq M$ is said to be a cut (abbr. $I \subseteq_e M$) if $a < b \in I \rightarrow a \in I$ and $I$ is closed under addition $+$ and multiplication $\cdot$.
- $\text{Cod}(M) = \{X \subseteq M \mid X \text{ is } M\text{-finite}\}$, where $M$-finite set is a set coded by an element in $M$ (by means of the usual binary coding).
- for $Z \in \text{Cod}(M)$, $|Z|$ denotes the internal cardinality of $Z$ in $M$.
- for $I \subseteq_e M$, $\text{Cod}(M/I) := \{X \cap I \mid X \in \text{Cod}(M)\}$.

**Proposition**

*If $I \subseteq_e M$, then $I$ is a $\Sigma_0$-elementary substructure of $M$.***
There are several important types of cuts.

**Theorem (exponentially closed cut, Simpson/Smith)**

Let $M \models \text{EFA}$, and let $I \subseteq_e M$. Then the following are equivalent.

1. $(I, \text{Cod}(M/I)) \models \text{WKL}_0^*.$
2. $I$ is closed under exp.

**Theorem (semi-regular cut)**

Let $M \models \text{EFA}$, and let $I \subseteq_e M$. Then the following are equivalent.

1. $(I, \text{Cod}(M/I)) \models \text{WKL}_0.$
2. $I$ is semi-regular, i.e., if $X \in \text{Cod}(M)$ and $|X| \in I$, then $X \cap I$ is bounded in $I$. 
Let $M \models \text{EFA}$, and let $I \subsetneq_e M$. Then the following are equivalent.

1. $(I, \text{Cod}(M/I)) \models \text{ACA}_0$.

2. $I$ is strong, i.e., if for any $a > I$ for any $b \in I$ and for any $f : [0, a]^3 \to b$ coded in $M$, there exists $Y \subseteq [0, a]$ such that $Y$ is $f$-homogeneous and $Y \cap I$ is unbounded in $I$.

These combinatorial characterization of cuts play key roles in the definition of indicators.
Let $T$ be a theory of second-order arithmetic. A $\Sigma_0$-definable function $Y : [M]^2 \rightarrow M$ is said to be an indicator for $T \supseteq \text{WKL}_0^*$ if

- $Y(x, y) \leq y$,
- if $x' \leq x < y \leq y'$, then $Y(x, y) \leq Y(x', y')$,
- $Y(x, y) > \omega$ if and only if there exists a cut $I \subseteq e M$ such that $x \in I < y$ and $(I, \text{Cod}(M/I)) \models T$.

(Here, $Y(x, y) > \omega$ means that $Y(x, y) > n$ for any standard natural number $n$.)

Example

- $Y(x, y) = \max\{n : \exp^n(x) \leq y\}$ is an indicator for WKL$_0^*$.
- $Y(x, y) = \max\{n : \text{any } f[[x, y]]^n \rightarrow 2 \text{ has a homogeneous set } Z \subseteq [x, y] \text{ such that } |Z| > \min Z\}$ is an indicator for ACA$_0^*$. 
Basic properties of indicators

**Theorem**

If $Y$ is an indicator for a theory $T$, then for any $n \in \omega$, 

$$T \vdash \forall x \exists y Y(x, y) \geq n.$$ 

**Theorem**

If $Y$ is an indicator for a theory $T$, then, $T$ is a $\Pi^0_2$-conservative extension of EFA + \{\forall x \exists y Y(x, y) \geq n \mid n \in \omega\}.

Let $F^Y_n(x) = \min\{y \mid Y(x, y) \geq n\}$.

**Theorem**

If $Y$ is an indicator for a theory $T$ and $T \vdash \forall x \exists y \theta(x, y)$ for some $\Sigma^1$-formula $\theta$, then, there exists $n \in \omega$ such that 

$$T \vdash \forall x \exists y < F^Y_n(x) \theta(x, y).$$
Let $T$ be a theory of second-order arithmetic. A $\Sigma_0$-definable function $Y : \text{Cod}(M) \to M$ is said to be a set indicator for $T \supseteq \text{WKL}_0^*$ if

1. $Y(F) \leq \max F$,
2. if $F \subseteq F'$, then $Y(F) \leq Y(F')$,
3. $Y(F) > \omega$ if and only if there exists a cut $I \subseteq M$ such that $\min F \in I < \max F$ and $(I, \text{Cod}(M/I)) \models T$, and $F \cap I$ is unbounded in $I$.

Note that if $Y$ is a set indicator, then $Y'(x, y) = Y([x, y])$ is an indicator function.

**Example**

$Y(F) = \max\{m : F \text{ is } m\text{-dense}(\text{RT}_2^2)\}$

is an indicator for $\text{WKL}_0 + \text{RT}_2^2$.

Actually, density notions provide set indicators for many theories.
A Ramsey-like-$\Pi^1_2$-formula is a $\Pi^1_2$-formula of the form

$$\forall f : [\mathbb{N}]^n \rightarrow k)(\exists Y)(Y \text{ is infinite } \land \Psi(f, Y))$$

where $\Psi(f, Y)$ is of the form $\forall G \subseteq \text{fin } Y)\Psi_0(f \upharpoonright [[0, \max G]_{\mathbb{N}}]^n, G)$ such that $\Psi_0$ is a $\Delta^0_0$-formula.

(Here, $n, k \in \omega$ or they are unbounded parameters.)

- In particular, $\text{RT}^n_k$ is a Ramsey-like-$\Pi^1_2$-statement where $\Psi(f, Y)$ is the formula “$Y$ is homogeneous for $f$”.
- Any $\Pi^1_2$-formula of the form $\forall X\exists Y \Theta(X, Y)$ where $\Theta$ is a $\Sigma^0_3$-formula is equivalent to a Ramsey-like formula over $\text{WKL}_0$.

A Ramsey-like statement has an indicator given by the density notion.
Definition (EFA, Density notion)

Given a Ramsey-like formula
\[ \Gamma = (\forall f : [\mathbb{N}]^n \rightarrow k)(\exists Y)(Y \text{ is infinite } \wedge \psi(f, Y)), \]

- \( Z \subseteq_{\text{fin}} \mathbb{N} \) is said to be 0-dense(\( \Gamma \)) if \(|Z|, \min Z > 2\),
- \( Z \subseteq_{\text{fin}} \mathbb{N} \) is said to be \((m + 1)\)-dense(\( \Gamma \)) if
  - (for any \( n, k < \min Z \) and) for any \( f : [[0, \max Z]]^n \rightarrow k \), there is an \( m\)-dense(\( \Gamma \)) set \( Y \subseteq Z \) such that \( \psi(f, Y) \) holds, and,
  - for any partition \( Z_0 \sqcup \cdots \sqcup Z_{\ell-1} = Z \) such that \( \ell \leq Z_0 < \cdots < Z_{\ell-1} \), one of \( Z_i \)'s is \( m\)-dense(\( \Gamma \)).

Note that “\( Z \) is \( m\)-dense(\( \Gamma \))” can be expressed by a \( \Delta_0 \)-formula.

Put \( Y_\Gamma(F) := \max\{m \mid F \text{ is } m\text{-dense}(\Gamma)\} \).

Theorem

\( Y_\Gamma \) is a set indicator for \( \text{WKL}_0 + \Gamma \).
Basic properties of indicators (review)

**Theorem**

*If $Y$ is an indicator for a theory $T$, then for any $n \in \omega$,*

\[ T \vdash \forall x \exists y Y(x, y) \geq n. \]

**Theorem**

*If $Y$ is an indicator for a theory $T$, then, $T$ is a $\Pi^0_2$-conservative extension of $\text{EFA} + \{\forall x \exists y Y(x, y) \geq n \mid n \in \omega\}$.*

Let $F^Y_n(x) = \min\{y \mid Y(x, y) \geq n\}$.

**Theorem**

*If $Y$ is an indicator for a theory $T$ and $T \vdash \forall x \exists y \theta(x, y)$ for some $\Sigma_1$-formula $\theta$, then, there exists $n \in \omega$ such that*

\[ T \vdash \forall x \exists y < F^Y_n(x) \theta(x, y). \]
Basic properties of set indicators

Theorem

If $Y$ is a set indicator for a theory $T$, then for any $n \in \omega$,

$$T \vdash \forall X \subseteq_{\text{inf}} \mathbb{N} \exists F \subseteq_{\text{fin}} X(Y(F) \geq n).$$

Theorem

If $Y$ is a set indicator for a theory $T$, then, $T$ is a $\tilde{\Pi}^0_3$-conservative extension of $\text{RCA}^*_0 + \{ \forall X \subseteq_{\text{inf}} \mathbb{N} \exists F \subseteq_{\text{fin}} X(Y(F) \geq n) \mid n \in \omega \}$. 

Theorem

If $Y$ is a set indicator for a theory $T$ and $T \vdash \forall X \subseteq_{\text{inf}} \mathbb{N} \exists F \subseteq_{\text{fin}} X\theta(F)$ for some $\Sigma_1$-formula $\theta$, then, there exists $n \in \omega$ such that

$$T \vdash \forall Z \subseteq_{\text{fin}} \mathbb{N}(Y(Z) \geq n \rightarrow \exists F \subseteq Z \theta(F)).$$
Some consequences ($\tilde{\Pi}_3^0$-part of $RT_2^2$...) 

- $WKL_0 + RT_2^2$ is a $\tilde{\Pi}_3^0$-conservative extension of $RCA_0^* + \{\forall X \subseteq \mathbb{N} \exists F \subseteq \text{fin} \ X(\text{F is } n\text{-dense}(RT_2^2)) \mid n \in \omega\}$. 
  \[(\equiv RCA_0 + \{nPH_2^2 \mid n \in \omega\})\]

- $WKL_0 + RT^2$ is a $\tilde{\Pi}_3^0$-conservative extension of $RCA_0^* + \{\forall X \subseteq \mathbb{N} \exists F \subseteq \text{fin} \ X(\text{F is } n\text{-dense}(RT^2)) \mid n \in \omega\}$.

- $ACA_0 + RT = ACA'_0$ is a $\Pi_1^1$-conservative extension of $RCA_0^* + \{\forall X \subseteq \mathbb{N} \exists F \subseteq \text{fin} \ X(\text{F is } n\text{-dense}(RT)) \mid n \in \omega\}$.

- $ACA_0 + HT(k)$ is a $\Pi_1^1$-conservative extension of $RCA_0^* + \{\forall X \subseteq \mathbb{N} \exists F \subseteq \text{fin} \ X(\text{F is } n\text{-dense}(HT(k))) \mid n \in \omega\}$.

- $ACA_0 + HT$ is a $\Pi_1^1$-conservative extension of $RCA_0^* + \{\forall X \subseteq \mathbb{N} \exists F \subseteq \text{fin} \ X(\text{F is } n\text{-dense}(HT)) \mid n \in \omega\}$.

- ... 

Here, HT denotes Hindman’s theorem.
Some consequences (Generalized Parsons theorem)

Since $\omega^n$-largeness implies $n$-density($0 = 0$), i.e., a density notion for $\text{WKL}_0$, we have the following.

Theorem (Generalized Parsons theorem)

Let $\psi(F)$ be a $\Sigma^0_1$-formula with exactly the displayed free variables. Assume that for a given Ramsey-like statement $\Gamma$,

\[
\text{WKL}_0 + \Gamma \vdash \forall X \subseteq \mathbb{N}(X \text{ is infinite} \rightarrow \exists F \subseteq_{\text{fin}} X \psi(F)).
\]

Then, there exists $n \in \omega$ such that

\[
\text{WKL}_0 + \Gamma \vdash \forall Z \subseteq_{\text{fin}} \mathbb{N}(Z \text{ is } n\text{-dense}(\Gamma) \rightarrow \exists F \subseteq Z \psi(F)).
\]

In particular,

\[
\text{WKL}_0 \vdash \forall X \subseteq \mathbb{N}(X \text{ is infinite} \rightarrow \exists F \subseteq_{\text{fin}} X \psi(F)).
\]

Then, there exists $n \in \omega$ such that

\[
\text{WKL}_0 \vdash \forall Z \subseteq_{\text{fin}} \mathbb{N}(Z \text{ is } \omega^n\text{-large} \rightarrow \exists F \subseteq Z \psi(F)).
\]
Density with the base $\text{ACA}_0$

### Definition (EFA, Density notion with the base $\text{ACA}_0$)

Given a Ramsey-like formula

$$\Gamma = (\forall f : [\mathbb{N}]^n \rightarrow k)(\exists Y)(Y \text{ is infinite } \land \Psi(f, Y)),$$

- $Z \subseteq \text{fin } \mathbb{N}$ is said to be $0$-dense'$^\prime'$(\(\Gamma\)) if $|Z| > 4$, $\min Z > 2$,
- $Z \subseteq \text{fin } \mathbb{N}$ is said to be $(m + 1)$-dense'$^\prime'$(\(\Gamma\)) if
  - (for any $n, k < \min Z$ and) for any $f : [[0, \max Z]]^n \rightarrow k$, there is an $m$-dense'$^\prime'$(\(\Gamma\)) set $Y \subseteq Z$ such that $\Psi(f, Y)$ holds, and,
  - for any partition $f : [Z]^3 \rightarrow \ell$ such that $\ell < \min Z$ there is an $m$-dense'$^\prime'$(\(\Gamma\)) set $Y \subseteq Z$ which is $f$-homogeneous.

Put $Y'_\Gamma(F) := \max\{m \mid F \text{ is } m\text{-dense}'(\Gamma)\}$.

### Theorem

$Y'_\Gamma$ is a set indicator for $\text{ACA}_0 + \Gamma$.

With $\text{ACA}_0$, one can always characterize the $\Pi^1_1$-part of $\Gamma$. 
Definition (EFA, Density notion with the base WKL\(_0^*\))

Given a Ramsey-like formula
\[
\Gamma = (\forall f : [\mathbb{N}]^n \to k)(\exists Y)(Y \text{ is infinite } \wedge \Psi(f, Y)),
\]
- \(Z \subseteq_{\text{fin}} \mathbb{N}\) is said to be 0-dense\(^*\)(\(\Gamma\)) if \(Z \neq \emptyset\),
- \(Z \subseteq_{\text{fin}} \mathbb{N}\) is said to be \((m + 1)\)-dense\(^*\)(\(\Gamma\)) if
  - (for any \(n, k < \min Z\) and) for any \(f : [[0, \max Z]]^n \to k\), there is an \(m\)-dense\(^*\)(\(\Gamma\)) set \(Y \subseteq Z\) such that \(\Psi(f, Y)\) holds, and,
  - \(Z \setminus [0, \exp(\min Z)]\) is \(m\)-dense\(^*\)(\(\Gamma\)).

Put \(Y^*_\Gamma(F) := \max\{m \mid F \text{ is } m\text{-dense}^*(\Gamma)\}\).

Theorem

\(Y^*_\Gamma\) is a set indicator for WKL\(_0^* + \Gamma\).
Conservation theorems for $\text{RT}_{k}^{n}$ and $\text{HT}(k)$ over $\text{WKL}_{0}^{*}$

- $\text{WKL}_{0}^{*} + \text{RT}_{k}^{n}$ is a $\tilde{\Pi}_{3}^{0}$-conservative extension of $\text{RCA}_{0}^{*} + \{ \forall X \subseteq_{\inf} \mathbb{N} \exists F \subseteq_{\fin} X(F \text{ is } n\text{-dense}^{*}(\text{RT}_{k}^{n})) \mid n \in \omega \}$.
  $\equiv \text{RCA}_{0}^{*}$

- $\text{WKL}_{0}^{*} + \text{RT} = \text{ACA}_{0}'$ is a $\tilde{\Pi}_{3}^{0}$-conservative extension of $\text{RCA}_{0}^{*} + \{ \forall X \subseteq_{\inf} \mathbb{N} \exists F \subseteq_{\fin} X(F \text{ is } n\text{-dense}^{*}(\text{RT})) \mid n \in \omega \}$.

- $\text{WKL}_{0}^{*} + \text{HT}(k)$ is a $\tilde{\Pi}_{3}^{0}$-conservative extension of $\text{RCA}_{0}^{*} + \{ \forall X \subseteq_{\inf} \mathbb{N} \exists F \subseteq_{\fin} X(F \text{ is } n\text{-dense}^{*}(\text{HT}(k))) \mid n \in \omega \}$.
  $\equiv \text{RCA}_{0}^{*}$

- $\text{WKL}_{0}^{*} + \text{HT} = \text{ACA}_{0} + \text{HT}$ is a $\tilde{\Pi}_{3}^{0}$-conservative extension of $\text{RCA}_{0}^{*} + \{ \forall X \subseteq_{\inf} \mathbb{N} \exists F \subseteq_{\fin} X(F \text{ is } n\text{-dense}^{*}(\text{HT})) \mid n \in \omega \}$.

Thus, $\text{WKL}_{0}^{*} + \text{RT}_{k}^{n}$ and $\text{WKL}_{0}^{*} + \text{HT}(k)$ are very weak, while $\text{WKL}_{0}^{*} + \text{RT}$ and $\text{WKL}_{0}^{*} + \text{HT}$ are not.
Thank you!
