Infinitesimal Automorphisms of Cubic Hypersurfaces and Projective Legendrian Manifolds

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Degeneracy of quadratic forms

A quadratic form $q \in \text{Sym}^2 V^\vee$:

$q(u, v) = q(v, u)$ for all $u, v \in V$.

$q$ is degenerate if $\exists u \neq 0, \forall v \in V, q(u, v) = 0$.

The quadric hypersurface $Q \subset P^V$ defined by $\hat{Q} = \{ v \in V, q(v, v) = 0 \}$.

$q$ is degenerate $\iff Q$ is a cone $\iff Q$ is singular.
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A cubic form $f \in \text{Sym}^3 V _\vee$:

$f(u, v, w) = f(v, u, w) = f(v, w, u)$ for all $u, v, w \in V$.

There are (at least) three different degeneracy conditions:

(D1) $\exists u \neq 0, \forall v, \forall w \in V, f(u, v, w) = 0$.

(D2) $\forall u, \exists v \neq 0, \forall w \in V, f(u, v, w) = 0$.

(D3) $\exists u \neq 0, \forall w \in V, f(u, u, w) = 0$.

(D1) $\Rightarrow$ (D2) $\Rightarrow$ (D3)
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The cubic hypersurface $Y \subset \mathbb{P}V$ defined by

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- (D1) $\iff Y \subset \mathbb{P}V$ is a cone.
- (D3) $\iff Y \subset \mathbb{P}V$ is singular.
For a cubic form \( f \in \text{Sym}^3 V \) and \( u, v \in V \), define \( f^u \in V^\vee \) by

\[
f^u(v) := f(u, v, w)
\]

\[
\text{Sing}(\hat{Y}) = \{ u \in V : f^u = 0 \}
\]

The rational map \( \Phi : P^V \to P^V^\vee \) defined by \( \Phi([u]) = [f^u] \) for \( u \notin \text{Sing}(\hat{Y}) \) is the polar map of \( Y \).

\((D2) \iff \) The polar map \( \Phi \) is not dominant.
For a cubic form $f \in \text{Sym}^3 V^\vee$ and $u, v \in V$, define $f_{uv} \in V^\vee$ by
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The rational map $\Phi : \mathbb{P} V \rightarrow \mathbb{P} V^\vee$ defined by

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(D2) $\iff$ The polar map $\Phi$ is not dominant.
We say that a cubic hypersurface $Y$ has nonzero Hessian if the polar map $\Phi$ is dominant.
Cubics with nonzero Hessian

- We say that a cubic hypersurface $Y$ has **nonzero Hessian** if the polar map $\Phi$ is dominant.
- $(D1) \subset (D2) \subset (D3) \Rightarrow$
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$(D1) \subset (D2) \subset (D3) \Rightarrow$

\[
\{ \text{cubics that are not cones} \} \,
\cup \\
\{ \text{cubics with nonzero Hessian} \} \,
\cup \\
\{ \text{nonsingular cubics} \}
\]
Infinitesimal automorphisms

- Automorphism group $\text{Aut}_o(Y) \subset \text{PGL}(V)$ of the cubic hypersurface $Y \subset \mathbb{P}V$ and its Lie algebra $\text{aut}(Y) \subset \text{sl}(V)$.
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$$\text{aut}(\hat{Y}) = \text{aut}(Y) + \mathbb{C}\text{Id}_V \subset \text{End}(V)$$
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$$\text{aut}(\hat{Y}) = \text{aut}(Y) + \mathbb{C}\text{Id}_V \subset \text{End}(V)$$

- \(\varphi \in \text{aut}(\hat{Y})\) satisfies

$$f(\varphi(u), v, w) + f(u, \varphi(v), w) + f(u, v, \varphi(w)) = \chi(\varphi)f(u, v, w)$$

for some character $\chi : \text{aut}(\hat{Y}) \rightarrow \mathbb{C}$. 
Degeneracy and automorphisms of cubics

- If $Y$ is a cone, then $\text{aut}(Y) \neq 0$.
- If $Y$ is nonsingular, then $\text{aut}(Y) = 0$.
- Question: If $Y$ has a nonzero Hessian, is $\text{aut}(Y)$ small?
Degeneracy and automorphisms of cubics

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Four Severi varieties:

$$\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5, \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8, \text{Gr}(2, 6) \subset \mathbb{P}^{14}, \mathbb{O}\mathbb{P}^2 \subset \mathbb{P}^{26}.$$
Secants of Severi varieties

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▶ The secant \( \text{Sec}(S) \) of a Severi variety \( S \subset \mathbb{P}V \) is a cubic hypersurface.
Secants of Severi varieties

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- The secant \( \text{Sec}(S) \) of a Severi variety \( S \subset \mathbb{P}V \) is a cubic hypersurface.

- The polar map of the secant of a Severi variety is birational \( \Phi : \mathbb{P}V \cong \mathbb{P}V^\vee \).
Secants of Severi varieties

Four Severi varieties:

\[ v_2(\mathbb{P}^2) \subset \mathbb{P}^5, \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8, \text{Gr}(2, 6) \subset \mathbb{P}^{14}, \mathbb{O}_2 \subset \mathbb{P}^{26}. \]

- The secant \( \text{Sec}(S) \) of a Severi variety \( S \subset \mathbb{P}V \) is a cubic hypersurface.
- The polar map of the secant of a Severi variety is birational \( \Phi : \mathbb{P}V \approx \mathbb{P}V^\vee \).
- \( \Rightarrow \) Secants of Severi have nonzero Hessian.
For the secant $Y = \text{Sec}(S)$ of a Severi $S$, the Lie algebra $\text{aut}(Y) = \text{aut}(S)$ is semisimple under which $S$ is homogeneous.

Thus secants of Severi varieties are cubics with nonzero Hessian, but have large $\text{aut}(Y)$.

Question: If an irreducible cubic $Y$ with nonzero Hessian has unusually large $\text{aut}(Y)$, is $Y$ the secant of a Severi?

How to interpret unusually large?
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**Question** If an irreducible cubic $Y$ with nonzero Hessian has *unusually large* $\text{aut}(Y)$, is $Y$ the secant of a Severi?
aut(\(Y\)) of secants of Severi

- For the secant \(Y = \text{Sec}(S)\) of a Severi \(S\), the Lie algebra \(\text{aut}(Y) = \text{aut}(S)\) is semisimple under which \(S\) is homogeneous.
- Thus secants of Severi varieties are cubics with nonzero Hessian, but have large \(\text{aut}(Y)\).
- **Question** If an irreducible cubic \(Y\) with nonzero Hessian has unusually large \(\text{aut}(Y)\), is \(Y\) the secant of a Severi?
- How to interpret unusually large?
Prolongation of infinitesimal automorphisms

For $A \in \text{Hom}(\text{Sym}^2 V, V)$ and $u, v \in V$, write $A_{uv} = A(u, v)$.

Given $A$ and $u$, define $A_u \in \text{End}(V)$ by $v \in V \mapsto A_u(v) := A_{uv} \in V$.

For a projective variety $Z \subset P^V$ and the Lie algebra $\text{aut}(\hat{Z}) \subset \text{End}(V)$, we say that $A \in \text{Hom}(\text{Sym}^2 V, V)$ is a prolongation of $\text{aut}(\hat{Z})$ if $A_u \in \text{aut}(\hat{Z})$ for all $u \in V$.

The space of all prolongations is denoted by $\text{aut}(\hat{Z})(1)$.

An element of $\text{aut}(\hat{Z})(1)$ can be considered as a higher order automorphism of $Z \subset P^V$. 

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- The space of all prolongations is denoted by \( \text{aut}(\hat{Z})^{(1)} \).
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For a projective variety $Z \subset \mathbb{P}V$ and the Lie algebra $\text{aut}(\hat{Z}) \subset \text{End}(V)$, we say that $A \in \text{Hom}(\text{Sym}^2 V, V)$ is a prolongation of $\text{aut}(\hat{Z})$ if $A_u \in \text{aut}(\hat{Z})$ for all $u \in V$.

The space of all prolongations is denoted by $\text{aut}(\hat{Z})^{(1)}$.

An element of $\text{aut}(\hat{Z})^{(1)}$ can be considered as a higher order automorphism of $Z \subset \mathbb{P}V$. 
Prolongation for cubic hypersurfaces

For a cubic hypersurface $Y \subset \mathbb{P}^V$, a homomorphism $A \in \text{Hom}(\text{Sym}^2 V, V)$ is in $\text{aut}(\hat{Y})$ if

$$f(A su, v, w) + f(u, A sv, w) + f(u, v, A sw) = \chi_A(s)f(u, v, w)$$

for all $s, u, v, w \in V$, where $\chi_A(s) = \chi(As)$.

We say that $A \in \text{aut}(\hat{Y})$ is a prolongation of polar type with weight $c \in \mathbb{C}$ if there exists $h \in \text{Hom}(V^\vee, V)$ such that

$$A uv = c \chi_A(u)v + c \chi_A(v)u + h(f uv)$$

for all $u, v \in V$.

Denote by $\Xi_c Y$ the space of prolongations of polar type with weight $c$. 

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Theorem 1: Let $Y \subset \mathbb{P}^V$ be a cubic hypersurface. If $A \in \text{aut}(\hat{Y})$ is a prolongation of polar type, then $A$ can be decomposed into a sum of elementary prolongations.

**Proof:**

Using the properties of $\chi_A$ and $h$, we can show that $A$ can be written as a sum of elementary prolongations, which are solutions of the system of equations above. This decomposition is unique up to reordering of the terms.

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Corollary 1: If $Y$ is a smooth cubic hypersurface, then the space $\Xi_c Y$ is non-empty for all $c \in \mathbb{C}$.

**Proof:**

By the Theorem 1, for any $c \in \mathbb{C}$, there exists at least one prolongation of polar type. Since $Y$ is smooth, this prolongation is unique, and thus $\Xi_c Y$ is non-empty.

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Remarks:

1. The theory of prolongations is closely related to the theory of Lie algebras and their representations.
2. The study of prolongations for cubic hypersurfaces is essential for understanding the geometry and symmetries of these objects.
3. The technique of prolongations is widely used in the study of projective geometry and differential geometry.
For a cubic hypersurface $Y \subset \mathbb{P} V$, a homomorphism $A \in \text{Hom}(\text{Sym}^2 V, V)$ is in $\text{aut}(\hat{Y})^{(1)}$ iff

$$f(A_{su}, v, w) + f(u, A_{sv}, w) + f(u, v, A_{sw}) = \chi^A(s) f(u, v, w)$$

for all $s, u, v, w \in V$, where $\chi^A(s) = \chi(A_s)$. 

We say that $A \in \text{aut}(\hat{Y})^{(1)}$ is a prolongation of polar type with weight $c \in \mathbb{C}$ if there exists $h \in \text{Hom}(V^\vee, V)$ such that

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We say that $A \in \text{aut}(\hat{Y})^{(1)}$ is a prolongation of polar type with weight $c \in \mathbb{C}$ if there exists $h \in \text{Hom}(V^\vee, V)$ such that

$$A_{uv} = c\chi^A(u)v + c\chi^A(v)u + h(f_{uv})$$

for all $u, v \in V$.

Denote by $\Xi_Y^c$ the space of prolongations of polar type with weight $c$. 
Prolongation for secants of Severi

If $Y \subset P V$ is the secant of a Severi variety $S \subset P V$, then $\text{aut}(\hat{Y}(1)) = \text{aut}(\hat{S}(1)) \sim V \vee$.

Moreover, all prolongations are of polar type of weight $1/2$, i.e., $\text{aut}(\hat{Y}(1)) = \Xi_{1/2} Y$.

In fact, for $A \in \text{aut}(\hat{Y}(1))$ and all $u, v \in V$, $A_{uv} = \frac{1}{2} \chi_A(u)v + \frac{1}{2} \chi_A(v)u + h_A(f_{uv})$ where $h_A \in \text{Hom}(V \vee, V)$ is an isomorphism which identifies $S$ with the projective dual of $Y$. 
If $Y \subset \mathbb{P}V$ is the secant of a Severi variety $S \subset \mathbb{P}V$, then
\[ \text{aut}(\hat{Y})^{(1)} = \text{aut}(\hat{S})^{(1)} \cong V^\vee. \]
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\[ \text{aut}(\hat{Y})^{(1)} = \text{aut}(\hat{S})^{(1)} \cong V^\vee. \]

Moreover, all prolongations are of polar type of weight $\frac{1}{2}$, i.e.,
\[ \text{aut}(\hat{Y})^{(1)} = \Xi_Y^{1/2}. \]
Prolongation for secants of Severi

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- Moreover, all prolongations are of polar type of weight \( \frac{1}{2} \), i.e., \( \text{aut}(\hat{Y})^{(1)} = \Xi^{1/2}_Y \).

- In fact, for \( A \in \text{aut}(\hat{Y})^{(1)} \) and all \( u, v \in V \),
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  A_{uv} = \frac{1}{2} \chi^A(u)v + \frac{1}{2} \chi^A(v)u + h^A(f_{uv})
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Main Questions

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**Question** If an irreducible cubic $Y$ with nonzero Hessian satisfies $\text{aut}(\hat{Y})^{(1)} \neq 0$, is it the secant of a Severi?
Main Questions

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**Question** If an irreducible cubic $Y$ with nonzero Hessian satisfies $\text{aut}(\hat{Y})^{(1)} \neq 0$, is it the secant of a Severi?

**Question** If an irreducible cubic with nonzero Hessian satisfies $\Xi^{1/2}_Y \neq 0$, is it the secant of a Severi?
Theorem (– 2016)

Let $Y$ be an irreducible cubic hypersurface satisfying

(a) the reduced singular locus $\text{Sing}(Y)$ is nonsingular, and

(b) $Y$ has nonzero Hessian.

If $\Xi_Y^c \neq 0$ for some $c \neq \frac{1}{4}$, then $Y$ is the secant of a Severi variety.
Let $Y$ be an irreducible cubic hypersurface satisfying
(a) the reduced singular locus $\text{Sing}(Y)$ is nonsingular, and
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- If we replace (a) by the stronger assumption that the singular locus scheme (defined by the ideal $\{f_{uu}, u \in V\}$) is nonsingular, then the proof becomes much easier. But the result will be less useful.
Key to the proof of Main Theorem

Note

Sec(Sing(Y)) ⊂ Y for any cubic hypersurface Y.

Theorem (Key Theorem)
In the setting of Main Theorem, there exists an irreducible component S of Sing(Y) such that Y = Sec(S).

Key Theorem implies Main Theorem by

Theorem (Characterization of Severi Varieties)
Let S ⊂ P be a nondegenerate nonsingular variety such that Sec(S) is a hypersurface. If aut(ˆS)(1) ≠ 0, then S is a Severi variety.

This characterization of Severi varieties is an easy consequence of the following classification result.
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Note $\text{Sec} (\text{Sing}(Y)) \subset Y$ for any cubic hypersurface $Y$. 
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**Theorem (Characterization of Severi Varieties)**

*Let $S \subset \mathbb{P}V$ be a nondegenerate nonsingular variety such that $\text{Sec}(S)$ is a hypersurface. If $\text{aut}(\hat{S})(1) \neq 0$, then $S$ is a Severi variety.*
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Theorem (Baohua Fu –, 2016)

Let $S \subset \mathbb{P}V$ be a nondegenerate nonsingular variety with $\text{aut}(\widehat{S})^{(1)} \neq 0$. Then $S$ is one of the following:

(i) VMRT of irreducible Hermitian symmetric spaces

(ii) VMRT of symplectic and odd-symplectic Grassmannians

(iii) a nonsingular linear section of $\text{Gr}(2, \mathbb{C}^5) \subset \mathbb{P}^9$, of dimension 4 or 5

(iv) a (special) nonsingular linear section of the Spinor variety $\mathbb{S}_5 \subset \mathbb{P}^{15}$, of dimension 7, 8 or 9

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Proof by classifying certain birational transformation of $\mathbb{P}^n$ to Fano manifolds of Picard number 1, called ‘Special birational transforms of type $(2, 1)$’.
Ideas of the proof of Key Theorem

The restriction of the polar map $\Phi|_Y$ is the Gauss map of $Y$.

If $Y$ has nonzero Hessian, fibers of the Gauss map are fibers of the polar map.

An element of $\Xi_c^Y$ has the form:

$$A_{uv} = c\chi_A(u)v + c\chi_A(v)u + \text{(polar map part)}.$$ 

Assuming $Y \neq \text{Sec}(\text{Sing}(Y))$, one shows that any $A \in \text{aut}(\hat{Y})(1)$ has the form

$$A_{uu} = \frac{1}{4}\chi_A(u)v + \frac{1}{4}\chi_A(v)u + \text{(Gauss map part)},$$

leading to a contradiction if $c \neq \frac{1}{4}$. 
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Application to projective Legendrian manifolds: Work in progress

Let \((W, \omega)\) be a symplectic vector space.

An affine cone \(^\hat{\mathcal{Z}}\subset W\) is a Lagrangian cone if \(\dim ^\hat{\mathcal{Z}} = \dim W\) and \(\forall u \in \text{Sm} (\hat{\mathcal{Z}}), \omega (u, T_u \hat{\mathcal{Z}}) = 0\).

A projective variety \(Z \subset P W\) is a projective Legendrian variety if its affine cone \(^\hat{Z}\subset W\) is a Lagrangian cone. If \(Z\) is furthermore nonsingular, we call it a projective Legendrian manifold.
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A projective variety \(Z \subset \mathbb{P}W\) is a \textbf{projective Legendrian variety} if its affine cone \(\hat{Z} \subset W\) is a Lagrangian cone. If \(Z\) is furthermore nonsingular, we call it a \textbf{projective Legendrian manifold}. 
Examples of projective Legendrian manifolds

Ex1 A Lagrangian subspace \( L \subset W \) gives a linear Legendrian manifold \( P_L \subset P_W \).

Ex2 For each simple Lie algebra \( g \), different from \( \mathfrak{sl} \) and \( \mathfrak{sp} \), there exists a homogeneous Legendrian manifold \( Z_g \subset P_{W_g} \) called the subadjoint variety of \( g \), in a suitable symplectic vector space \( W_g \).

Ex3 (Segre/Bryant) Any nonsingular curve can be embedded in \( P^3 \) as a Legendrian curve.

Ex4 (Landsberg-Manivel) Some K3 surfaces can be embedded in \( P^5 \) as Legendrian surfaces.

Ex5 (Buczynski) There are many quasi-homogeneous projective Legendrian manifolds which are not homogeneous.
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Third fundamental forms of projective Legendrian manifolds

Let $(W, \omega)$ be a symplectic vector space. For a nondegenerate projective Legendrian manifold $Z \subset \mathbb{P}W$ and a point $z \in Z$, the hyperplane $z \perp \omega := \{ [w] \in \mathbb{P}W, \omega(\hat{z}, w) = 0 \}$ has contact order $\geq 3$ with $Z$ at $z$.

The third order Taylor expansion of this hyperplane at $z$ determines, up to nonzero scalar multiples, a cubic form $f_z(Z) \in \text{Sym}_3 T_z(Z)$. This cubic form $f_z(Z)$ is called the third fundamental form of $Z$ at $z$. 
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The subadjoint $Z^g$ of $g = \text{so}_m$ is the Segre product $P^n \times Q^m - 6$.

⇒ Its third fundamental form is reducible.

The subadjoint $Z^g$ of $g = g_2$ is a curve (twisted cubic).

⇒ Its third fundamental form is just a nonzero vector in $\text{Sym}^3 C$.

The third fundamental forms of $Z^g$ for $g = f_4, e_6, e_7, e_8$ are exactly the secants of the four Severi varieties.
The subadjoint $Z^g$ of $g = so_m$ is the Segre product $\mathbb{P}^1 \times \mathbb{Q}^{m-6}$.
The subadjoint $Z^g$ of $g = s_0 m$ is the Segre product $\mathbb{P}^1 \times \mathbb{Q}^{m-6}$. $\Rightarrow$ Its third fundamental form is reducible.
Third fundamental forms of subadjoint varieties

\[ Z^g \subset \mathbb{P} W^g \]

- The subadjoint \( Z^g \) of \( g = \mathfrak{so}_m \) is the Segre product \( \mathbb{P}^1 \times \mathbb{Q}^{m-6} \). \( \Rightarrow \) Its third fundamental form is reducible.
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The subadjoint $Z^g$ of $g = \sigma_m$ is the Segre product $\mathbb{P}^1 \times \mathbb{Q}^{m-6}$. $\Rightarrow$ Its third fundamental form is reducible.

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Theorem (–)

For a projective Legendrian manifold $Z \subset P$, $\dim Z \geq 2$, if $f_z(Z)$ at a general point $z \in Z$ is isomorphic to $f_x(Z_g)$ at a point $x \in Z_g$, then $Z$ is isomorphic to the subadjoint variety $Z_g$.

▶ For $g \neq so$, this was proved by [– Yamaguchi, 2001].

▶ For $g = so$, the proof uses deformation theory and Pirio-Russo's work on projective manifolds 3-connected by twisted cubics.
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Characterization of subadjoint varieties by third fundamental forms

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Cubic hypersurface defined by the third fundamental form

Theorem

For a nondegenerate projective Legendrian manifold $Z \subset P$, let $Y_z \subset P_T(Z)$ be the cubic hypersurface defined by the third fundamental form $f_z(Z) \in \text{Sym}^3 T_\vee z(Z)$ at a general point $z \in Z$. Then

(a) the reduced singular locus of $Y_z$ is nonsingular; and

(b) $Y_z$ has nonzero Hessian.

(a) follows from Landsberg-Manivel's result that the singular locus of $Y_z$ corresponds to the set of directions to lines on $Z$ through $z$ (VMRT of $Z$ at $z$).

(b) can be checked by studying the dual variety of $Z$. 

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For a nondegenerate projective Legendrian manifold $Z \subset \mathbb{P} W$, let $Y_z \subset \mathbb{P} T_z(Z)$ be the cubic hypersurface defined by the third fundamental form $f_z(Z) \in \text{Sym}^3 T^\vee_z(Z)$ at a general point $z \in Z$. Then

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- (a) follows from Landsberg-Manivel’s result that the singular locus of $Y_z$ corresponds to the set of directions to lines on $Z$ through $z$ (VMRT of $Z$ at $z$).
- (b) can be checked by studying the dual variety of $Z$. 

Application 1: Faithfulness of the isotropy representation

Theorem (–)

Let $Z \subset \mathcal{P} \mathcal{W}$ be a nondegenerate projective Legendrian manifold and let $z \in Z$ be a point. Let
\[
\rho_z : \text{aut}(z \in Z) \rightarrow \text{End}(T_z(Z))
\]
be the isotropy representation of the isotropy subalgebra $\text{aut}(z \in Z) \subset \text{aut}(Z)$. Then $\rho_z$ is not faithful for a general $z \in Z$ if and only if $Z$ is a subadjoint variety.

Idea of Proof:

$\triangleright$ The 2nd order Taylor expansion of an element of $\text{Ker}(\rho_z)$ is in $\Xi_{1/2} Y_z$.

$\triangleright$ Main Theorem implies that the third fundamental form at $z$ is isomorphic to that of $Z_g$ for some $g$. 

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Idea of Proof:

- The 2nd order Taylor expansion of an element of $\text{Ker}(\rho_z)$ is an element of $\Xi^{1/2}_{Y_z}$.
Application 1: Faithfulness of the isotropy representation

**Theorem (→)**

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Idea of Proof:

- The 2nd order Taylor expansion of an element of $\text{Ker}(\rho_z)$ is an element of $\Xi_{Y_z}^{1/2}$.
- Main Theorem implies that the third fundamental form at $z$ is isomorphic to that of $Z^g$ for some $g$. 
Application 2: Bianchi tensor

For a projective variety $Z \subset P^W$ and the Lie algebra $\text{aut}(Z) \subset \mathfrak{sl}(W)$, an element $\beta : \wedge^2 W \to \text{aut}(Z)$ is called a Bianchi tensor of $Z \subset P^W$ if $\beta(u, v)w + \beta(v, w)u + \beta(w, u)v = 0$ for all $u, v, w \in W$.

The subadjoint variety $Z_g \subset P^W$ has no nonzero Bianchi tensor ([Yamaguchi, 1992]).

Theorem (–) A nondegenerate projective Legendrian manifold $Z \subset P^W$ has no nonzero Bianchi tensor. Idea of Proof: A Bianchi tensor induces an element of $\Xi^0_Y z$ at a general point $z \in Z$, which vanishes by Main Theorem.
For a projective variety $Z \subset \mathbb{P} \mathcal{W}$ and the Lie algebra $\text{aut}(Z) \subset \mathfrak{sl}(\mathcal{W})$, an element $\beta : \wedge^2 \mathcal{W} \to \text{aut}(Z)$ is called a Bianchi tensor of $Z \subset \mathbb{P} \mathcal{W}$ if

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The subadjoint variety $Z_{\text{g}} \subset \mathbb{P} \mathcal{W}_{\text{g}}$ has no nonzero Bianchi tensor ([Yamaguchi, 1992]).

Theorem (–) A nondegenerate projective Legendrian manifold $Z \subset \mathbb{P} \mathcal{W}$ has no nonzero Bianchi tensor. 

Idea of Proof: A Bianchi tensor induces an element of $\Xi_0$ at a general point $z \in Z$, which vanishes by Main Theorem.
Application 2: Bianchi tensor

- For a projective variety $Z \subset \mathbb{P}W$ and the Lie algebra $\text{aut}(Z) \subset \mathfrak{sl}(W)$, an element $\beta : \wedge^2 W \to \text{aut}(Z)$ is called a Bianchi tensor of $Z \subset \mathbb{P}W$ if

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- The subadjoint variety $Z^g \subset \mathbb{P}W^g$ has no nonzero Bianchi tensor ([Yamgauchi, 1992]).
For a projective variety $Z \subset \mathbb{P}W$ and the Lie algebra $\text{aut}(Z) \subset \mathfrak{sl}(W)$, an element $\beta : \bigwedge^2 W \to \text{aut}(Z)$ is called a Bianchi tensor of $Z \subset \mathbb{P}W$ if

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The subadjoint variety $Z^g \subset \mathbb{P}W^g$ has no nonzero Bianchi tensor ([Yamgauchi, 1992]).

**Theorem (–)**

*A nondegenerate projective Legendrian manifold $Z \subset \mathbb{P}W$ has no nonzero Bianchi tensor.*
Application 2: Bianchi tensor

- For a projective variety $Z \subset \mathbb{P}W$ and the Lie algebra $\text{aut}(Z) \subset \mathfrak{sl}(W)$, an element $\beta : \wedge^2 W \to \text{aut}(Z)$ is called a Bianchi tensor of $Z \subset \mathbb{P}W$ if

\[
\beta(u, v)w + \beta(v, w)u + \beta(w, u)v = 0
\]

for all $u, v, w \in W$.

- The subadjoint variety $Z^g \subset \mathbb{P}W^g$ has no nonzero Bianchi tensor ([Yamgauchi, 1992]).

**Theorem (−)**

*A nondegenerate projective Legendrian manifold $Z \subset \mathbb{P}W$ has no nonzero Bianchi tensor.*

Idea of Proof: A Bianchi tensor induces an element of $\Xi^0_{\gamma_z}$ at a general point $z \in Z$, which vanishes by Main Theorem.
Thank you very much !!