Iitaka conjecture and abundance for 3-folds in char $= p > 5$

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A task of algebraic geometry is to classify smooth projective varieties.

\[ \dim X = 1: \]
\[ g = h^0(X, \omega_X) \sim \mathcal{M}_g: \text{moduli of curves of genus } g. \]

\[ \dim X = 2: \]
(1) **Difficulty**: there are too many surfaces \( X_n \) birational to \( X \), i.e., \( \exists \) open sets \( U_n \subset X_n \) isomorphic to an open set of \( X \).
(2) Reasonable to classify varieties up to birational equivalence.
(3) Find a good model from a birationally equivalent class in a standard way.
(4) **Strategy**: choosing the “minimal” variety \( X \), which admits no blowing down maps.
Solution in dimension 2
For a surface $X$, one can go a minimal model program (MMP for short)

\[
\text{blowing down } X \to X_1 \to X_2 \to \ldots \to X^m = \tilde{X}
\]

to get a smooth birational model $\tilde{X}$ such that either
(1) $\tilde{X} \cong \mathbb{P}^2$ or is ruled by rational curves, or
(2) $K_{\tilde{X}}$ is nef and is semi-ample.
In the latter case $\tilde{X}$ is called the \textbf{minimal model} of $X$. 
Kodaira dimension

Kodaira dimension is an invariant to refine the classification.

- Kodaira dimension: \( \kappa(X) := \max_{m>0} \dim \phi_{|mK_X|}(X) \), which can be \(-\infty, 0, 1, \cdots, \dim X\).

- For surfaces, in case (1), \( \kappa = -\infty \).

- In case (2), \( \kappa = 0, 1, 2 \), and the case \( \kappa = 0 \) is completely classified; when \( \kappa = 0 \) Iitaka fibration of \( \bar{X} \) is an elliptic fibration; when \( \kappa = 2 = \dim X \) we say \( X \) is of general type.

- A divisor \( D \) is said to be semi-ample, if for sufficiently divisible \( N > 0 \), \( |ND| \) has no base point.

- In Case (2), \( K_{\bar{X}} \) is semi-ample. We say “abundance” holds for surfaces.
Minimal model program

\( \dim X \geq 3: \) similar story happens.
From \( X \), after a sequence of birational modification (divisorial contraction, flips), the so-called “MMP” one expects to get a “minimal” variety \( \bar{X} \) such that either

- \( \bar{X} \) is a Mori fiber space, or
- \( K_{\bar{X}} \) is nef.

Two central problems of MMP include

- Existence of minimal models in dimension \( n \) (\( E_n \)): Can MMP run and terminate?
- Abundance in dimension \( n \) (\( A_n \)): if \( K_X \) is nef, is \( K_X \) semi-ample?
Log-minimal model program

To study MMP by induction on dimensions, varieties with boundary \((X, B)\) should be considered. Analogous theory is called \textbf{log MMP}.

Central problems of log-MMP include

- Existence of log minimal models in dimension \(n\) (log \(E_n\))
- Abundance in dimension \(n\) for lc minimal models (log \(A_n\))

Remark: Mild singularities, say, singularities below should be permitted in higher dimension to make MMP run. For a pair \((X, B)\), taking a log smooth resolution \(f : Y \to X\) and writing that

\[
K_Y = f^*(K_X + B) + \sum_i a_i E_i,
\]

if \(a_i \geq -1\) (> \(-1\)), we say \((X, B)\) is \textbf{log-canonical} (Kawamata log terminal).
Over complex numbers, remarkable progresses were made by many brilliant mathematicians: Mori, Kawamata, Shokurov, Reid, Kollar, Viehweg, BCHM, Fujino, Gongyo....

The following has been proved

- \( \log E_n \) and \( \log A_n \) when \( B \) is big or \( K_X + B \) is big [BCHM, 2010];
- \( \log E_3 \) and \( \log A_3 \) [Miyaoka, Kawamata, KMM, 1990s].
Progresses in characteristic $p$

In dimension 2, 

- existence of minimal models and abundance are implied by Bombieri-Mumford’s classification [BM, 1970s],
- similar results are true for (semi-)log canonical surfaces [Tanaka, 2014-15].

In dimension 3 and char $p > 5$, 

- existence of minimal models of klt pairs $(X, B)$ [Hacon, Xu, Birkar, 2013-15];
- abundance for klt pairs when either $K_X + B$ or $B$ is big [Birkar and Xu, 2015].

Problem

Prove abundance for a minimal klt pair $(X, B)$ of 3-fold with $\nu(K_X + B) = 0, 1, 2$. 
Approach in characteristic zero: \( q(X) > 0 \)

Two completely different methods are employed according to \( q(X) \), i.e., whether \( X \) has Albanese map.

1. If \( q(X) > 0 \), i.e., \( X \) has non-trivial Albanese map \( a_X : X \to A_X \), one can show abundance by using

Theorem: Kawamama, Viehweg, 1980

Let \( X \) be a smooth projective variety over \( \mathbb{X} \) of maximal Albanese map. If \( \kappa(X) = 0 \), then \( X \) is birational to an abelian variety.

Iitaka Conjecture

Let \( f : X \to Y \) be a fibration between two smooth projective varieties over \( \mathbb{C} \), with \( \dim X = n \) and \( \dim Y = m \). Then

\[
C_{n,m} : \kappa(X) \geq \kappa(Y) + \kappa(X_{\bar{\eta}}).
\]

In dimension 3, it is proven by K-V (1980s).
Approach: the case \( q(X) = 0 \)

(2) The case \( q(X) = 0 \) was treated by [Miyaoka, Kawamata, 1990s].

The approach is as follows

Step 1: Prove the non-vanishing of \( H^0(X, nK_X) \). Miyaoka applied Riemann-Roch formula,

\[
\chi(Y, \rho^*\mathcal{O}(nK_X)) = \frac{2n^3 - 3n^2}{12}K_X^3 + \frac{n}{12}K_X \cdot (K_X^2 + \rho^*c_2(Y)) + \chi(\mathcal{O}_X)
\]

the following are needed

P1 generic positivity of \( \Omega_X \) (implying \( c_2(X) \geq 0 \));

P2 vanishing theorems;

P3 Donaldson’s results on stable bundles: poly-stable vector bundles on a surface with vanishing Chern classes are induced by representation of \( \pi_1^{alg} \);

P4 the fundamental group of a terminal germ \((X, 0)\) is finite.
Approach continued: the case $q(X) = 0$ and log abundance

Step 2: Prove $\kappa(X) > 0$ if $K_X \sim 0$, which implies abundance. The following are needed [Miyaoka and Kawamata, 1980s-90s]

P5 log abundance of surfaces (which has been proven by Tanaka in char $p > 0$)

P6 for case $\nu(K_X) = 1$, infinitesimal deformation of canonical divisors in 3-folds

P7 for case $\nu(K_X) = 2$, log terminal singularities are quotient singularities in codimension two, generic positivity results on orbifolds, vanishing theorems

(3) Log abundance was proved by reducing to Mori fiber spaces [Keel, Matsuki, McKernan, 1990s], where the key point is

P8 canonical bundle formula
We have additional difficulties in positive characteristics.

**P9** For a fibration of smooth varieties $f : X \to Y$, $X_{\bar{\eta}}$ may be singular, in particular it is non-reduced if $f$ is inseparable. So it is much more difficult to show Iitaka conjecture.

**P10** For a minimal variety $X$, $X$ may be uniruled. Then generic positivity of $\Omega^1_X$ fails.

Remark:

- We don’t worry about P2, P5 by [Tanaka, 2015] and P6 by [Totaro, 2009] in positive characteristics.
- P4 has been studied by [Carvajal-Rojas, Schwede and Tucker, 2015] for $F$-regular singularities.
- P1 and P8 are widely open and of great importance.
Iitaka conjecture for 3-folds

Theorem: Ejiri, Birkar, Chen, -, 2015-16

Assume \( \text{char } k = p > 5 \). Let \((X, B)\) be a klt 3-folds and \( f : X \to Y \) a separable fibration. Then

1. \( C_{3,1} \) is true when
   (1.1) \((X_\eta, B_\eta)\) is klt, \( p \not\mid \text{ind}(B_\eta) \) and \( \kappa(X_\eta, K_{X_\eta} + B_\eta) = 0, 2 \); or
   (1.2) \( \kappa(X_\eta, K_{X_\eta} + B_\eta) = 1 \), and \( K_{X_\eta} + B_\eta \) induces an elliptic fibration.

2. \( C_{3,2} \) is true when
   (2.1) \( g(X_\eta) > 0 \), or
   (2.2) \( Y \) is not uni-ruled, \( K_Y \) is big, and \( \kappa(X_\eta, K_{X_\eta} + B_\eta) = 1 \).
Theorem: -, 2016

Let $X$ be a $\mathbb{Q}$-factorial, projective, non-uniruled 3-fold, over an algebraically closed field of characteristic $p > 5$. Let $B$ be an effective $\mathbb{Q}$-divisor on $X$. Assume that

1. $(X, B)$ is a minimal klt pair; and
2. the Albanese map $a_X : X \to A_X$ is non-trivial.

Then $K_X + B$ is semi-ample.
Sketch of the proof

(1) Reduce to showing that either $\kappa(K_X + B) \geq 1$ or $K_X + B \sim_{\mathbb{Q}} 0$.

(2) If the Albanese map $a_X : X \rightarrow A_X$ is separable, then the Stein factorization of $a_X$ induces a separable fibration $f : X \rightarrow Y$. We can prove abundance by use of MMP, subadditivity of Kodaira dimensions, geometry of varieties with $\kappa(X) = 0$.

(3) If the Albanese map $a_X : X \rightarrow A_X$ is inseparable, then we have a foliation $\mathcal{F} = \mathcal{L}^\perp \subset T_X$ where $\mathcal{L}$ is the saturation of the image of the natural homomorphism $a_X^* \Omega^1_{A_X} \rightarrow \Omega^1_X$. By replacing $X$ with $X/\mathcal{F}$ repeatedly, we can finally obtain a variety whose Albanese map is separable, then show that $\kappa(X) \geq 1$ by induction. Details will be explained below.
How to treat inseparable maps?

We explain the idea to prove that $\kappa(X) \geq 1$ if the Albanese map $a_X : X \to A_X$ is inseparable, $\dim a_X(X) = \dim X - 1$ and $X$ is not uniruled.

(1) Let $\mathcal{L}$ denote the saturation of the image of the natural homomorphism $a_X^* \Omega^1_{A_X} \to \Omega^1_X$. Then $\mathcal{L}$ is generically globally generated, $\operatorname{rank} \mathcal{L} \leq n - 2$, and $h^0(X, \mathcal{L}) \geq h^0(A_X, \Omega^1_{A_X}) \geq n - 1$, which implies that
\[ h^0(X, \det \mathcal{L}) \geq 2. \]

(2) We get a natural foliation $\mathcal{F} = \mathcal{L}^\perp \subset T_X$ of rank $\geq 2$ and a quotient map $\rho : X \to X_1 = X/\mathcal{F}$. Then we have
\[ \kappa(X) \geq \kappa(X_1), \] and if $\kappa(X_1) = 0$ then $\kappa(X) \geq 1$.

If $a_{X_1} : X_1 \to A$ is separable, then we are done by applying $C_{n, n-1}$. 
(3) Let $X'$ be the normalization of the reduction of $X^{(1)} \times_{Z^{(1)}} Z$. Then

We can show the multiplicity of the geometric fiber of $X_1 \rightarrow Z_1$ decrease strictly if $a_{X_1}$ is not inseparable.
Further questions

To prove abundance for 3-folds, the following probably should be concerned according to Miyaoka, Kawamata and KMM

- generic positivity of $\Omega_X$ for non-uniruled 3-fold;
- a reasonable canonical bundle formula;
- local fundamental group of klt pairs;
- a method to treat uniruled cases.
On canonical bundle formula

In char = 0, for a klt pair \((X, B)\) and a \(K_X + B\)-trivial fibration \(f : X \to Y\), we have a klt pair \((Y, B_Y)\) and the canonical bundle formula (Ambro, 2004):

\[
K_X + B \sim_{\mathbb{Q}} f^*(K_Y + B_Y).
\]

In positive characteristics, Das and Schwede (Ejiri) prove that if \((X_\eta, B_\eta)\) is F-split (F-pure) then there exists an effective (pseudo-effective) divisor \(B_Y\) on \(Y\) such that

\[
K_X + B \sim_{\mathbb{Q}} f^*(K_Y + B_Y).
\]

If not assuming \((X_\eta, B_\eta)\) is F-split or F-pure, the results above is not true. One can get an example by considering \(P_C(V)\) where \(V\) is a semi-stable but not strongly semi-stable. vector bundle of rank two.
Thank you!


S. Ejiri and L. Zhang *Iitaka’s C_{n,m} conjecture for 3-folds in positive characteristic*, arXiv: 1604.01856.


