Infinite Dimensional Affine Processes.

Dedicated to Professor Ito

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Introduction

Affine processes contain the standard linear diffusion equations with constant volatility, like the Ornstein-Uhlenbeck process, but also some nonlinear diffusion equations, like the Feller process [4], used later in many applications, [4], [1]. Basically, an affine process has a characteristic function which is affine with respect to the initial condition. We want to consider infinite dimensional affine processes. We use a general framework to study linear infinite dimensional stochastic equations. We consider next a class of non-linear problems. We also show that the framework is adequate to studying infinite-dimensional affine processes.
Feller Process

The Feller process is defined by the equation

\[ dy = k(\theta - y)dt + c\sqrt{y}dw \]

If we consider the Riccati equation

\[-q' + kq = \frac{1}{2}c^2q^2, \quad q(T) = \lambda \]

Provided this equation has a global solution on \(0, T\) we have the formula

\[ E \exp y(T)\lambda = E \exp y(0)q(0) \exp k\theta \int_0^T q(t)dt \]
Linear Random Functionals

Let $\Phi$ be a Hilbert space, $\Phi'$ its dual. A 2nd order L.R.F. on $\Phi'$ is a family $\zeta_{\phi^*}(\omega)$ such that

$$\phi^* \rightarrow \zeta_{\phi^*}(\omega) \in \mathcal{L}(\Phi'; L^2(\Omega, A, P)).$$

We write

$$E\zeta_{\phi^*} = \langle m, \phi^* \rangle$$

$$E\zeta_{\phi^*} \zeta_{\tilde{\phi}^*} - E\zeta_{\phi^*} E\zeta_{\tilde{\phi}^*} = \langle \Gamma \phi^*, \tilde{\phi}^* \rangle.$$

$m$ mathematical expectation, $\Gamma$ covariance operator.
If $\Gamma$ is nuclear

$$\zeta_{\phi_*} = \langle \phi_*, \zeta(\omega) \rangle$$

$$\zeta(\omega) = \sum_i \phi_i \zeta J \phi_i$$

where $J$ is the isometry from $\Phi$ to $\Phi'$, and $\phi_i$ is an orthonormal basis of $\Phi$; $J \phi_i$ is an orthonormal basis of $\Phi'$. The random variable $\zeta$ satisfies

$$E\zeta^2 = \sum_i \langle J^* \Gamma J \phi_i, \phi_i \rangle$$
Assume

\[ \Phi = L^2(0, T; E) \]

where \( E \) is a Hilbert space, whose dual is denoted by \( E' \). So

\[ \Phi' = L^2(0, T; E') \]

A LRF on \( \Phi' \) is denoted by \( \xi_{e^*}(.)(\omega) \), where \( e^*(.) \) is an element of \( \Phi' \). We shall be particularly interested in LRFs on \( \Phi' \) with a covariance operator given by

\[ < \Gamma e^*(.), \tilde{e}^*(.) > = \int_0^T < Q(t)e^*(t), \tilde{e}^*(t) > dt \quad (1) \]

where \( Q(.) \in L^\infty(0, T; \mathcal{L}(E'; E)) \).
Gaussian LRF

A gaussian LRF is such that

\[ \zeta_{\phi_*} \text{ is gaussian } \forall \phi_* \]

We shall in particular concentrate on the case

\[ \Phi' = L^2(0, T; E') \]

with the covariance operator defined by \( I \), and 0 mean. We then consider a filtration \( \mathcal{E}^t \) and assume that

\[ \xi_{e_*} \mathbb{1}_{0,s} \text{ is } \mathcal{E}^t \text{ measurable } , \forall e_* \in E', \forall s \leq t \] (2)

\[ \xi_{e_*} \mathbb{1}_{s,t} \text{ is independent of } \mathcal{E}^s , \forall e_* \in E', \forall s \leq t \]
When

\[ \mathcal{E}^t = \sigma(\xi_{e*} I_{0,s}, \forall e* \in E', \forall s \leq t) \]

then the second property follows from the gaussian assumption and the non correlation of \( \xi_{e*} I_{s,t} \) with the variables generating \( \mathcal{E}^s \). The stochastic processes \( \xi_{e*} I_{0,t} \), up to an equivalence, form a family of Wiener processes, indexed by \( e* \). Moreover

\[ E\xi_{e*} I_{0,t} \xi_{\tilde{e}*} I_{0,s} = \int_0^{\min(t,s)} Q(\tau) d\tau e*, \tilde{e}*> \]
Consider a basis $e_{*,i}$ of $E'$, call

$$w_i(t) = \xi_{e_{*,i}} \mathbb{I}_{(0,t)}$$

Then, it is easy to check that

$$\xi_{e_*, \mathbb{I}_{0,t}} = \sum_i w_i(t) < e_*, e_i >$$

the convergence taking place in $L^2(\Omega, A, P)$, for any $t$. 

Set

\[ e_i = J^{-1}e_{\star, i} \]

where \( J \) denotes the isomorphism from \( E \) to \( E' \). Next define

\[ w(t) = \sum_i w_i(t)e_i \]

This is a formal sum in \( E \).
Note that

\[ E\| \sum_{i=1}^{N} w_i(t)e_i \|^2 = \sum_{i=1}^{N} E(w_i(t))^2 = \]

\[ = \sum_{i=1}^{N} \int_{0}^{t} < Q(\tau)e_{*,i}, e_{*,i} > d\tau \]

So the formal limit converges in \( L^2(\Omega, A, P; E) \), if one has the nuclear property

\[ \sum_{i=1}^{\infty} \int_{0}^{t} < Q(\tau)e_{*,i}, e_{*,i} > d\tau < \infty \]
Identification

In the case when the assumptions (2) hold, we have constructed a family of Wiener processes $w_i(t)$. Now if $e_*(.) \in L^2(0, T; E')$ we have the identification

**Proposition 1.**

$$\xi_{e*}(.) = \sum_i \int_0^T < e_*(t), e_i > \, dw_i(t)$$

(3)
Generalizing Da Prato-Zabczyk [1], we call the formal sum

$$w(t) = \sum_i w_i(t)e_i$$  \hspace{1cm} (4)

a cylindrical Wiener process, with covariance operator $Q(.)$. We define the generalized stochastic integral

$$\int_0^T < e_*(t), dw(t) > = \sum_i \int_0^T < e_*(t), e_i > dw_i(t) = \xi e_*(.)$$  \hspace{1cm} (5)

which is well defined as an element of $L^2(\Omega, \mathcal{A}, P)$. 
Generalized stochastic integrals

Now the formal sum \( w(t) \) converges in a bigger space than \( E \). Indeed take any sequence \( \alpha_i \geq 0 \), with \( \sum_i \alpha_i = 1 \) and let us define

\[
E_1 = \{ e = \sum_i \lambda_i e_i \mid \sum_i \lambda_i^2 \alpha_i < \infty \} \tag{6}
\]

then we have the

**PROPOSITION 2.**

\[
w(t) \in L^\infty(0, T; L^2(\Omega, \mathcal{A}, P; E_1)) \tag{7}
\]
It follows that the stochastic integral (5) is well defined whenever \( e_\ast(.) \) belongs to \( L^2(0, T; E_1') \). Note that
\[
E_1' \subset E', \text{ with continuous injection}
\]
So the generalized stochastic integral is an extension of the ordinary stochastic integral, with integrands in \( L^2(0, T; E') \). However, the choice of \( E_1 \) is arbitrary, and does not play any role in the definition of \( \xi_{e_\ast(.)} \).
Moreover

\[ E \int_{0}^{T} < e_*(t), dw(t) > = 0 \]  \hspace{1cm} (8)

\[ E(\int_{0}^{T} < e_*(t), dw(t) >)^2 = \]

\[ E \int_{0}^{T} < Q(\tau)e_*(\tau; \omega), e_*(\tau; \omega) > d\tau. \]  \hspace{1cm} (9)
All properties of stochastic integrals extend **Proposition 3**.

\[
E \left[ \int_0^T < e_*(\tau) \mathbb{I}_{s,t}(\tau), dw(\tau) > \right] = 0 \quad (10)
\]

\[
E \left[ \left( \int_0^T < e_*(\tau) \mathbb{I}_{s,t}(\tau), dw(\tau) > \right)^2 \right] = \quad (11)
\]

\[
E \left[ \int_s^t < Q(\tau) e_*(\tau; \omega), e_*(\tau; \omega) > d\tau \right] = \quad (11)
\]
Consider the stochastic process

\[ I(t; \omega) = \int_0^T < e_\tau(\tau) \mathbb{I}_{0,t}(\tau), dw(\tau) > \]

**Proposition 4.** *Up to an equivalence, \( I(t) \) is a continuous process, and an \( \mathcal{E}^t \) martingale.*
If $\tau_1, \tau_2$ are $\mathcal{E}^t$ stopping times, then

$$E\left[ \int_0^T \mathbf{1}_{\tau_1,\tau_2}(t) \, dw(t) \right] |\mathcal{E}^{\tau_1}] = 0$$

$$E\left[ \left( \int_0^T \mathbf{1}_{\tau_1,\tau_2}(t) \, dw(t) \right)^2 \right] |\mathcal{E}^{\tau_1}] =$$

$$E\left[ \int_{\tau_1}^{\tau_2} \mathbf{1}_{\tau_1,\tau_2}(t) \, dw(t) \right] |\mathcal{E}^{\tau_1}] =$$

$$E\left[ \int_{\tau_1}^{\tau_2} Q(t)e_*(t) \, dt \right] |\mathcal{E}^{\tau_1}]$$
Finally, the stochastic integral can be extended to adapted processes such that

\[
\int_0^T \| e_\star(t) \|^2 dt < +\infty, \text{ a.s. .}
\]

We shall note

\[
I(t) = \int_0^t < e_\star(\tau), dw(\tau) >
\]
Ito’s Formula

A scalar stochastic process $\beta(t)$, continuous, adapted to $\mathcal{E}^t$ has an Ito differential whenever

$$\beta(t) = \beta_0 + \int_0^t \alpha(\tau) d\tau + \int_0^t < e_\ast(\tau), dw(\tau) > .$$

(12)

$\beta_0 \ \mathcal{E}^0$ measurable

(13)

$$E|\beta_0|^2 < +\infty.$$
Ito’s Formula

\[ \alpha(t) \text{ is adapted and } E \int_0^T \left| \alpha(t) \right|^2 dt < +\infty. \quad (14) \]

\[ e_\ast(t; \omega) \text{ is adapted with values in } E' \]

\[ E \int_0^T \left| e_\ast(t) \right|^2 dt < +\infty \quad (15) \]
THEOREM 1. Assume (12) to (15). Let $\psi(x, t)$ be a $C^{2,1}$ function, then the process $\psi(\beta(t), t)$ has an Ito differential given by

$$\psi(\beta(t), t) = \psi(\beta(0), 0) + \int_0^t \left[ \frac{\partial \psi(\beta(\tau), \tau)}{\partial \tau} ight] d\tau$$

$$+ \psi'(\beta(\tau), \tau) \alpha(\tau) + \frac{1}{2} \psi''(\beta(\tau), \tau) < Q(\tau)e_*(\tau), e_*(\tau) > d\tau$$

$$+ \int_0^t \psi'(\beta(\tau), \tau) < e_*(\tau), dw(\tau) >$$

(16)

Extensions are natural for functions $\psi(x, t)$ with $x \in \mathbb{R}^n$. 
Linear Evolution Equations

Consider the natural framework for linear evolution equations introduced by J.L. LIONS [2]. We start with a triple of Hilbert spaces, with continuous embedding

\[ V \subset H \subset V'. \]

Let us consider a family of linear operators

\[ A(\cdot) \in L^\infty(0, T; \mathcal{L}(V; V')) \] (17)

\[ \langle A(t)v, v \rangle \geq \alpha ||v||^2, \alpha > 0. \] (18)
Let $E$ be another Hilbert space, and

$$B(.) \in L^\infty(0, T; \mathcal{L}(E; V')).$$  \hspace{1cm} (19)$$

We consider two L.R.F. $\zeta_h(\omega)$ on $H$, and $\xi_{e_\ast}(\cdot)(\omega)$ on $L^2(0, T; E')$, with covariance operators $P_0$ and

$$Q(.) \in L^\infty(0, T; \mathcal{L}(E'; E))$$

We assume that

$$\xi_{e_\ast}(\cdot)(\omega) = \int_0^T < e_\ast(t), dw(t) >$$ \hspace{1cm} (20)
Recall that $w(t)$ is the cylindrical Wiener process

$$w(t) = \sum_i w_i(t)e_i$$

and has values in $E_1$. We want next to define the cylindrical process

$$\int_0^t B(s)dw(s)$$
For any $v(.) \in L^2(0, T; V)$, we define

$$\int_0^T < v(t), B(t)dw(t) >= \int_0^T < B^*(t)v(t), dw(t) >$$

where the right hand side is well defined since

$$B^*(.)v(.) \in L^2(0, T; E')$$
Also

\[ \int_0^T < v(t), B(t)dw(t) >= \xi_{B^*(.)}v(.) = \]

= \sum_i \int_0^T < v(t), B(t)e_i > dw_i(t)
We have the estimate

\[
E \left( \int_0^T \langle v(t), B(t)d\omega(t) \rangle \right)^2 \, dt = \\
E \int_0^T \langle B(t)Q(t)B^*(t)v(t), v(t) \rangle \, dt
\]
In particular taking

\[ v(.) = v \mathbb{I}_{(0,t)}, v \in V \]

we obtain

\[ < v, \int_0^T B(s)dw(s) > = \int_0^T < v \mathbb{I}_{(0,t)}(\tau), B(\tau)dw(\tau) > \]

(21)
As usual, \( \int_0^t B(s)dw(s) \) can be viewed as an element of

\[
V_1 = \left\{ \sum_i \lambda_i v_i \mid \sum_i \lambda_i^2 \alpha_i < \infty \right\}
\]

where \( v_i \) is an orthonormal basis of \( V \). We just write

\[
\int_0^t B(s)dw(s) = \sum_i v_i < v_i, \int_0^t B(s)dw(s) >
\]
We state the
**Theorem 2.** Assume (20),(17),(18),(19). There exists a unique L.R.F. \( y_{\phi_*}(\omega) \) on \( L^2(0, T; V') \) and a unique family \( y_h(t; \omega) \) of L.R.F. on \( H \), such that

\[
y_h(t) \in L^\infty(0, T; \mathcal{L}(H; L^2(\Omega, \mathcal{A}, P)))
\]

\[
\int_0^T y_{\phi_*}(t; \omega) \, dt = y_{\phi_*}(\omega)
\]

(22)

with an extension of the left hand side equation from \( L^2(0, T; H) \)to \( L^2(0, T; V') \). Equation (23) holds.
\[ y_h(t) + \int_0^t y_{A^*}(\tau) h(\tau) d\tau = \zeta_h + \langle h, \int_0^t B(\tau) d\omega(\tau) \rangle \]

\[ \forall h \in V, \forall t, \text{ a.s.} \]

(23)
Moreover, let $p$ and $q_t$ be defined by

$$-p' + A^*(t)p = \phi_*(t), \quad p(T) = 0 \quad (24)$$

$$-q'_t + A^*(\tau)q_t = 0, \quad q_t(t) = h \quad (25)$$

then one has the relations

$$y_h(t) = \zeta_{q_t(0)} + \int_0^t < q_t(\tau), B(\tau)dw(\tau) > \quad (26)$$

$$y_{\phi_*} = \zeta_{p(0)} + \int_0^T < p(\tau), B(\tau)dw(\tau) > \quad (27)$$
Correlation Operator

The correlation operator is defined by

\[(\Pi(t)h, h') = E y_h(t) y_{h'}(t)\]  \hspace{2cm} (28)

It verifies

\[\Pi(\cdot) \in L^\infty(0, T; \mathcal{L}(H; H)), \Pi(t) \geq 0, \text{ self adjoint}\]
**Theorem 3.** \textit{and} 

If $\theta \in L^2(0, T; V), \theta' \in L^2(0, T; V')$

\[-\theta' + A^*\theta \in L^2(0, T; H)\]

then $\Pi \theta \in L^2(0, T; V), (\Pi \theta)' \in L^2(0, T; V')$ \hspace{1cm} (29)

$(\Pi \theta)' + \Pi(-\theta' + A^*\theta) + A\Pi \theta = BQB^*\theta$

$\Pi(0) = P_0.$
Nonlinear Evolution Equations

Consider

\[ H_1 = \{ h = \sum_i \lambda_i h_i \mid \sum_i (\lambda_i)^2 \alpha_i < \infty \} \]

then the solution of 23 satisfies

\[ y(t) = \sum_i y_{h_i}(t) h_i \in L^\infty(0, T; L^2(\Omega, \mathcal{A}, P; H_1)) \]

(30)
We can treat some natural nonlinearities. Consider

\[ g : H_1 \rightarrow H, \quad B : H_1 \rightarrow \mathcal{L}(E; H) \]

such that

\[
|g(h) - g(h')|_H \leq C|h - h'|_{H_1} \tag{31}
\]

\[
||B(h) - B(h')||_{\mathcal{L}(E;H)} \leq C|h - h'|_{H_1} \tag{32}
\]
We consider the problem

\[ y_h(t) + \int_0^t y_{A^*}(\tau)h(\tau) d\tau = \]

\[ \zeta_h + \int_0^t (h, g(y(\tau))) d\tau + \int_0^t < h, B(y(\tau))d\omega(\tau) > \]

(33)

\[ \forall h \in V, \forall t, \text{ a.s.} \]

where \( y(t) \) is given by [30].
Statement of results

**Theorem 4.** We make the assumptions of 2 and 31, 32. There exists a unique L.R.F. \( y_{\phi^*}(\omega) \) on \( L^2(0, T; V') \) and a unique family \( y_h(t; \omega) \) of L.R.F. on \( H \), such that

\[
y_h(t) \in L^\infty(0, T; \mathcal{L}(H; L^2(\Omega, \mathcal{A}, P)))
\]

solution of 33.
Affine Processes

We take

\[ g(h, t) = f(t) + G(t)h, \quad G(.) \in L^\infty(0, T; \mathcal{L}(H; H)) \]  

(34)

\[ B(h, t) \in L^\infty(0, T; \mathcal{L}(E; H)) \]

\[ B(h, t)Q(t)B^*(h, t) = K_0(t) + \sum_{i=1}^{\infty} K_i(t)(h, h_i) \]

\[ K_0(.), K_i(.) \in L^\infty(0, T; \mathcal{L}(H; H)), \text{ self-adjoint} \]  

(35)
We assume

\[ \sum_i \sup_{0 \leq t \leq T} \| K_i(t) \| \leq C \]

We can see that if we take in the definition of \( H_1 \),
\[ \alpha_i = \sup_{0 \leq t \leq T} \| K_i(t) \| \]
then \( B(h, t) \) definition extends to \( H_1 \). But we do not have the Lipschitz property 32. Note that \( B(h, t) \) may not be defined for all \( h \).
Evolution Equation

We want to study the equation

\[ y_h(t) + \int_0^t y_{A^*(\tau)} h(\tau) d\tau = (y_0, h) + \zeta_h + \]

\[ \int_0^t (f(\tau), h) d\tau + \int_0^t y_{G^*(\tau)} h(\tau) d\tau + \int_0^t < h, B(y(\tau)) d\omega(\tau) \]

(36)

\[ \forall h \in V, \forall t, \text{ a.s.} \]

We cannot solve (36) in a strong sense.
We will again use the transposition method, but with different $q_{t,h}$, and define the solution of (36) in a weak sense. We define $q_{t,h}(\tau), \tau \leq t$ as follows

$$-q' + A^*(\tau)q = \frac{1}{2} \sum_i (q, K_i(\tau)q)h_i + G^*(\tau)q$$

$$q(t) = h$$

(37)

We also define $\alpha_{t,h}(\tau)$ by

$$\alpha_{t,h}(\tau) = \int_{\tau}^{t} \left[ (f(s), q_{t,h}(s)) + \frac{1}{2} (q_{t,h}(s), K_0(s)q_{t,h}(s)) \right]$$

(38)
Study of the Riccati Equation

To fix the ideas, we take $t = T$ in 37 and we write $q(.) = q_{T,h}(.)$. Consider (see 18)

$$< A(t)v, v > - (G(t)v, v) \geq \alpha \|v\|^2 - \sup_{0 \leq t \leq T} \|G(t)\| \|v\|^2$$

Therefore, we may assume that

$$- < A(t)v, v > + (G(t)v, v) \leq \beta |v|^2, \forall v \in H, \beta \in R$$

(39)

Define also

$$C = \sup_{0 \leq t \leq T} \left( \sum_i \|K_i(t)\|^2 \right)^{\frac{1}{2}}$$

(40)
We shall make the following assumption

if $\beta > 0$, then
$$\frac{\exp \beta T - 1}{\beta} < \frac{1}{|h|C}$$

(41)

if $\beta \leq 0$, then either $\beta > |h|C$ or $T < \frac{1}{|h|C}$

This assumption is a condition of smallness of $T$ or $|h|$. At any rate it follows that, whatever be the sign of $\beta$ one has

$$\frac{\exp \beta(T - t) - 1}{\beta} < \frac{1}{C|h|}$$

(42)
THEOREM 5. Assume that the injection of $V$ into $H$ is compact and we make the assumption 41. Then the Riccati equation

$$-q' + A^*(t)q = \frac{1}{2} \sum_i (q, K_i(t)q) h_i + G^*(t)q$$

(43)

$$q(T) = h$$

has a solution $q$ in $L^2(0, T; V) \cap C^0([0, T]; H)$, $q' \in L^2(0, T; H)$. Moreover

$$|q(t)| \leq \exp \beta^+ T \frac{\beta|h|}{\beta - C|h|(\exp \beta T - 1)}$$

(44)
**Weak solution of 36**

**Theorem 6.** We make the assumption 41. We also assume that the assumption is valid with $|h| = 1$. There exists a unique family $y_h(t; \omega)$ of L.R.F. on $H$, such that

$$y_h(t) \in L^\infty(0, T; \mathcal{L}(H; L^2(\Omega, A, P)))$$

solution of 36 in a weak sense. The process $y_h(t)$ is given explicitly by the formula

$$E \exp y_h(t) = E \exp \zeta_{q_t,h}(0) \exp[(y_0, h) + \alpha_{t,h}(0)]$$

$$= \exp\left[\frac{1}{2}(P_0 q_{t,h}(0), q_{t,h}(0)) + (y_0, h) + \alpha_{t,h}(0)\right]$$

(45)
Example

Assume

\[ B(h) = \sqrt{(h, h_1 I)} \]

defined for \( h \) such that \((h, h_1) \geq 0\). Moreover

\[ < A(t)h, \tilde{h} > = ((h, \tilde{h})) , \forall h, \tilde{h}. \]

We write

\[ dw(x, t) = \sum_{k} h_k(x) dw_k(t) . \]
So we get the infinite set of equations

\[ dy_i(t) + \lambda_i y_i(t)dt = \sqrt{y_1}dw_i(t) \]

The first equation defines \( y_1 \) as a positive process provided the initial condition is positive. For the other equations the stochastic integral is given.
Another Example of Affine Process

We take here

\[ H = L^2(0, 1), \quad V = H_0^1(0, 1). \]

We will consider the stochastic partial differential equation

\[
dy - y'' \, dt = \sqrt{y} \sum_k h_k \, dw_k(t)\]

\[ y(0, t) = y(1, t) = 0 \]

\[ y(x, 0) = \zeta(x) \geq 0 \]

where \( y'' \) is the second partial derivative with respect to \( x \).
We can only give a weak meaning to the solution of (46). We check formally that the solution of

\[ dy - y'' \; dt = \sqrt{y + \sum_k h_k \, dw_k(t)} \]

is positive, provided that the initial condition is positive. Indeed, consider \( \frac{1}{2} (y^-)^2 \) then

\[ \frac{1}{2} E \, d \frac{d}{dt} \int (y^-)^2(x, t) \, dx = \int y'' \, y^- \, dx \]

from which it follows easily that \( (y^-)^2(x, t) = 0 \).
Weak solution

The solution will be defined through the Riccati equation

\[-\frac{\partial q}{\partial t} - q'' = \frac{1}{2} q^2\]

\[q(0, t) = q(1, t) = 0\]

\[y(x, T) = h(x)\]

(47)
If we perform a formal computation of $d \exp \int y(x, t) q(x, t) \, dx$ we obtain

$$\frac{\partial}{\partial t} E \exp \int y(x, t) q(x, t) \, dx = 0$$

which implies

$$E \exp \gamma_h(T) = E \exp \xi_{qh}(0) \quad (48)$$

with the usual notation.
Study of the Riccati equation

Testing the Riccati equation with $q$ we obtain

$$-\frac{1}{2} \frac{\partial}{\partial t} |q(t)|^2_H + \int (q')^2 dx = \frac{1}{2} \int q^3 dx.$$ 

Next we use

$$\int |q|^3 dx \leq 2 \|q\|^2_{H^1_0} + \frac{1}{8} |q(t)|^4_H$$

and we obtain easily the inequality

$$\frac{\partial}{\partial t} \frac{1}{|q(t)|^2} \leq \frac{1}{8}$$
Therefore we can assert the estimate

\[ |q(t)|^2 \leq \frac{8|h|^2}{8 - T|h|^2} \]

provided that \( T|h|^2 < 8 \). Under this smallness condition the relation 48 is well defined.
Linear Filtering

We consider the linear evolution equation \( \frac{\partial y_h}{\partial t} = 23 \), and the LRF \( y_h(t) \) represents the state of a dynamic system at time \( t \). This state does not take values in \( H \), but in \( H_1 \). However, coordinates on a basis of \( H \) are well defined.

We define the observation process also by a L.R.F. Let \( \mathcal{C}(.) \in L^\infty(0, T; \mathcal{L}(V; F)) \).
Consider a gaussian L.R.F. on $L^2(0, T; F')$, denoted $\eta_{f^*(\cdot)}(\omega)$, independent from $\zeta_h$ and $\xi_{e^*(\cdot)}$, with correlation operator $R(t)$

$$
E \eta_{f_1^*(\cdot)} \eta_{f_2^*(\cdot)} = \int_0^T < R(t)f_1^*(t), f_2^*(t)> \, dt
$$

$$
R(\cdot) \in L^\infty(0, T; \mathcal{L}(F'; F)), \ R^{-1}(\cdot) \in L^\infty(0, T; \mathcal{L}(F; F'))
$$

(49)
In a way similar to $\xi_{e^*}(.)$ we can write

$$
\int_0^T < f^* (t), db(t) > = \sum_i \int_0^T < f^* (t), e_i > db_i(t) = \eta f^*
$$

where $b_i(t)$ are Wiener processes defined by

$$
b_i(t) = \eta_{f^* ,i} \mathbb{1}_{(0,t)}
$$

where the $f^* ,i$ form an orthonormal basis of $F'$. (50)
The observation is the L.R.F. defined by

$$Z_{f^*}(.) = \int_0^T yC^*(t)f^*(t)(t)dt + \int_0^T < f^*(t), db(t) > .$$

(51)

Let

$$\mathcal{B} = \sigma(Z_{f^*}(.), f^* \in L^2(0, T; F'))).$$

It is the $\sigma$ algebra of observations .

We define a L.R.F. on $H$ by

$$\hat{y}_h(T) = E[y_h(T)|\mathcal{B}] .$$

(52)
Kalman Filter

**Theorem 7.** The conditional probability of

\[ y_{h_1}(T), \ldots, y_{h_m}(T) \]

given \( \mathcal{B} \) is a gaussian with conditional mean

\[ \hat{y}_{h_1}(T), \ldots, \hat{y}_{h_m}(T) \]

and conditional correlation

\[ E[(y_{h_i}(T) - \hat{y}_{h_i}(T))(y_{h_j}(T) - \hat{y}_{h_j}(T))|\mathcal{B}] = \]

\[ (P(T)h_i, h_j). \]

\( P(t) \) is deterministic.
Assume
\[ C(.) \in L^\infty(0, T; \mathcal{L}(H; F)) \]  
then \( \hat{y}_h(t) \) is solution of the Kalman filter

\[
\hat{y}_h(t) + \int_0^t \hat{y}(A^* + C^* R^{-1} CP)(\tau)h(\tau)\,d\tau = Z_{R^{-1} CP(.)h} \mathbb{1}_{(0,t)} \quad \forall h \in V, \forall t, \text{ a.s.} \]
The operator $P(t)$ is the solution of the Riccati equation

$$P(.) \in L^\infty(0, T; \mathcal{L}(H; H)), \geq 0, \text{ self adjoint}$$

If $\theta \in L^2(0, T; V), \theta' \in L^2(0, T; V')$

$$-\theta' + A^*\theta \in L^2(0, T; H)$$

then $P\theta \in L^2(0, T; V), (P\theta)' \in L^2(0, T; V')$

$$(P\theta)' + P(-\theta' + A^*\theta) + AP\theta + PC^*R^{-1}CP\theta = BQB^*\theta$$

$P(0) = P_0$
Innovation

Consider the L.R.F. on $L^2(0, T; F')$

$$I_{f_*(\cdot)}(\omega) = Z_{f_*(\cdot)}(\omega) - \int_0^T \hat{y}_C(t) f_*(t)(t) dt$$

It is called the innovation L.R.F. Then one has the following result

**Theorem 8.** $I_{f_*(\cdot)}(\omega)$ is $\mathcal{B}$ measurable, gaussian, with

$$E I_{f_*(\cdot)} = 0$$

$$E I_{f_1^1(\cdot)} I_{f_2^2(\cdot)} = \int_0^T < R(t) f_1^1(t), f_2^2(t) > dt$$

the same as the noise on the observation.
Bibliography

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