ON ITÔ’S ONE POINT EXTENSIONS OF MARKOV PROCESSES
Masatoshi Fukushima

In [3], Kiyoshi Itô considered a general Markov process $X$ for which a point $a$ in the state space $S$ is regular for itself and recurrent. He associated a Poisson point process $p$ taking values in the space of excursion paths around $a$ using the inverse local time at $a$ and showed that $X$ is determined by $p$ and the stopped process $X^0$ of $X$ after hitting $a$. He also proved that the characteristic measure $n$ of $p$ is determined by the transition function $\{p_t^0, t \geq 0\}$ of $X^0$ and a $\{p_t^0\}$-entrance law $\{\mu_t; t > 0\}$.

[3] was an outgrowth of Itô’s joint work [1] with Henry McKean which described the most general extensions of the Brownian motion on a half line. Hiroshi Tanaka witnessed how a draft of [1] was energetically produced while he was staying in MIT. In a lecture [2] delivered in 1969, Itô determined and constructed most general Markovian extensions of $X^0$ with discontinuous entry from $a$, in which case $\mu_t = kp^0_t$ for a $\sigma$-finite measure $k$ on $S \setminus \{a\}$.

In the same year, I constructed in [4] a conservative diffusion extension of the absorbing Brownian motion $X^0$ on a bounded domain $D \subset \mathbb{R}^d$ to its one point compactification $D \cup \{\infty\}$ using a regular Dirichlet form. This extension is simpler than the reflecting Brownian motion on $\overline{D}$ and sometimes useful. Recently Tanaka asked me its relation to [3] so that we wrote a joint paper [5]. All possible Markovian extensions of $X^0$ need to be described in terms of quantities intrinsic to $X^0$. The dependence of the entrance law on $X^0$ for continuous entry was not fully analyzed in [3]. In [5], [6], we restrict ourselves to the cases where $X^0$ is symmetric or has a weak dual with respect to a fixed excessive measure $m$ and we decide all possible one-point extensions of $X^0$ that preserve symmetry or duality. A key observation is in that the entrance law is then uniquely determined by $m$ together with the approaching probability to $a$ of the dual process of $X^0$.

An application of Itô’s extension to a time changed reflecting Brownian motion on the closure of a unbounded domain will also be discussed.

References