ON ITÔ’S ONE POINT EXTENSIONS OF MARKOV PROCESSES

Masatoshi Fukushima

Symposium in Honor of Kiyosi Itô:
Stochastic Analysis and Its Impact in Mathematics and Science, IMS, NUS

July 10, 2008
§1. Itô’s point processes and related problems

[IM] K. Itô and H. P. McKean, Jr., Brownian motions on a half line,
*Illinois J. Math.* 7(1963), 181-231

[I-1] K. Itô, Poisson point processes and their application to Markov processes,
Lecture note of Mathematics Department, Kyoto University (unpublished), September 1969

[I-2] K. Itô, Poisson point processes attached to Markov processes,
In [I-2], Kiyosi Itô showed the following.

\[ X = (X_t, P_x) \]: a standard Markov process on a state space \( E \).

\( a \): a point of \( E \) such that \( a \) is regular for itself;

\[ P_a(\sigma_a = 0) = 1, \quad \sigma_a = \inf\{t > 0 : X_t = a\} \]

\( \{\ell_t, t \geq 0\} \): a local time at \( a \), namely, a PCAF of \( X \) with support \( \{a\} \).

Its right continuous inverse is defined by

\[ \tau_t = \inf\{s : \ell_s > t\}, \quad \inf\emptyset = \infty. \]

Define the space \( W \) of excursion paths around \( a \) by

\[ W = \{w : [0, \infty) \to E, \ 0 < \sigma_a(w), \ w_t = a, \ \forall t \geq \sigma_a(w)\} \]

Define a \( W \)-valued point process (Itô’s point process)

\( p \) by

\[ D(p) = \{s : \tau_s > \tau_{s-}\} \]

\[ p_s(t) = \begin{cases} X_{\tau_{s-} + t} & t \in [0, \tau_{s} - \tau_{s-}) \\ a & t \geq \tau_{s} - \tau_{s-}, \end{cases} \quad s \in D(p). \]

If \( \ell_\infty < \infty \), then \( p_{\ell_\infty} \) is a non-returning excursion.

Assume additionally that \( a \) is a recurrent point;

\[ P_x(\sigma_a < \infty) = 1, \quad \forall x \in E, \]

then \( \{\tau_t\} \) is a subordinator and

\( p \) is a \( W^+ \)-valued Poisson point process under \( P_0 \).
Denote by \( n \) the characteristic measure of \( p \).
\( n \) is a \( \sigma \)-finite measure on \( W \). Let
\[
\mu_t(B) = n(w_t \in B, t < \sigma_a(w)) \quad t > 0, \quad B \in \mathcal{B}(E \setminus \{a\}).
\]

Let \( X^0 \) be the process obtained from \( X \) by stopping at time \( \sigma_a \) and \( \{p_0^t; t \geq 0\} \) be its transition function.

1. \( X \) is determined by \( X^0 \) and \( n \).
2. \( \int_U (1 - e^{-\sigma_a}) n(dw) \leq 1 \).
3. \( \{\mu_t; t > 0\} \) is a \( \{p_t^0\} \)-entrance law: \( \mu_t p_s^0 = \mu_{t+s} \)
4. \( n \) is Markovian with semigroup \( p_t^0 \) and entrance law \( \{\mu_t\} \):
\[
\int_W f_1(w(t_1)) f_2(w(t_2)) \cdots f_n(w(t_n)) n(dw) = \mu_{t_1} f_1 P_{t_2-t_1}^0 f_2 \cdots P_{t_n-t_{n-1}}^0 f_{n-1} P_{t_n-t_{n-1}}^0 f_n.
\]
5. When \( n(w_0 = a) = 0 \) (discontinuous entry), then \( \mu_t = kp_t^0 \) for some \( \sigma \)-finite measure \( k \) on \( E \setminus \{a\} \) with
\[
\int_{E \setminus \{a\}} \mathbb{E}_x \left[ 1 - e^{-\sigma_a} \right] k(dx) < \infty,
\]
and conversely any such measure gives rise to a jump-in extension of \( X^0 \).

Itô stated 5 explicitly in a unpublished lecture note [I-1] and, as an application, he determined all possible extensions of the absorbed diffusion process on a half line with exit but non-entrance boundary yielding a generalization of a part of his joint paper [IM] with McKean.

removed the recurrence condition for $a$ from [I-2], and proved that Itô’s point process $p$ is equivalent to an absorbed(stopped) Poisson point process in the following sense:

Decompose the excursion space as $W = W^+ + W^- + \{\partial\}$. There exists a $\sigma$-finite measure $\tilde{n}$ on $W$ such that

- $\tilde{n}$ is Markovian with transition function $\{p^0_t\}$
- $\tilde{n}(W^- \cup \{\partial\}) < \infty$
- Let $\tilde{p}$ the $W$-valued Poisson point process with characteristic measure $\tilde{n}$ and $\tilde{T}$ be the first occurrence time of $W^- \cup \{\partial\}$. $\{\tilde{p}_t\}$ is then equivalent to the stopped point process $\{\tilde{p}_{\wedge \tilde{T}}\}$. 

X is said to be of continuous entry from a if \( \tilde{n}(w(0) \neq a) = 0 \).

**Problem I** (Uniqueness): when and how is the measure \( \tilde{n} \) on \( W \) uniquely determined by the minimal process \( X^0 \) ?

**Problem II** (Construction): when and how does \( X^0 \) admit extensions \( X \) with continuous entry from \( a \) ?


T. S. Salisbury, Construction of right processes from excursions, *PTRF* **73** (1986), 351-367


These important articles have dealt with generalizations of [I-2] but the dependence and independence on \( X^0 \) of the involved quantities were not clearly separated.

give affirmative answers to Problem I, II under a symmetry or weak duality setting for $X$.

- If a pair of standard processes $X$, $\hat{X}$ is in weak duality with respect to an excessive measure $m$, then $\tilde{n}$ is uniquely determined by $X^0$, $\hat{X}^0$ and $m_0 = m|_{E\setminus\{a\}}$ up to non-negative parameters $\delta_0$, $\hat{\delta}_0$ of the killing rate at $a$.
  In particular, if $X^0$ is $m_0$-symmetric, then its symmetric extension with no sojourn nor killing at $a$ is unique.

- If $X^0$, $\hat{X}^0$ are in weak duality with respect to $m_0$ and with no killing inside $E \setminus \{a\}$ and approachable to $a$, then they admit a pair of duality preserving extensions $X$, $\hat{X}$ with continuous entry from $a$ (by a time reversion argument).
§2. Uniqueness statements from [CFY]

§2.1. Description of \( \mathfrak{n} \) by exit system

\( \mathfrak{n} \) can be described in terms of the exit system due to [Mai] B.Maisonneuve, Exit systems. *Ann. Probab.* 3 (1975), 399-411.

\[ X = (\Omega, X_t, \mathbb{P}_x) : \text{a right process on a Lusin space } E. \]

A point \( a \in E \) is assumed to be regular for itself.

\( \ell_t : \text{a local time of } X \text{ at } a \)

\( \{p^t_0; t \geq 0\} : \text{the transition function of } X^0 \text{ obtained from } X \text{ by killing at } \sigma_a \)

\( \Omega : \text{the space of all paths } \omega \text{ on } E_\Delta = E \cup \Delta \text{ which are} \)

\( \text{cadlag up to the life time } \zeta(\omega) \text{ and stay at the cemetery } \Delta \text{ after } \zeta. \)

\( X_t(\omega) : \text{t-th coordinate of } \omega. \)

The *shift operator* \( \theta_t \) on \( \Omega \) is defined by

\[ X_s(\theta_t \omega) = X_{s+t}(\omega), \ s \geq 0. \]

The *killing operator* \( k_t, \ t \geq 0, \) on \( \Omega \) defined by

\[ X_s(k_t \omega) = \begin{cases} X_s(\omega) & \text{if } s < t \\ \Delta & \text{if } s \geq t. \end{cases} \]

The *excursion space* \( W \) is specified by

\[ W = \{k_{\sigma_a} \omega : \omega \in \Omega, \sigma_a(\omega) > 0\}, \]

which can be decomposed as

\[ W = W^+ \cup W^- \cup \{\partial\} \]

with

\[ W^+ = \{w \in W : \sigma_a < \infty\}, \ W^- = \{w \in W : \sigma_a = \infty \text{ and } \zeta > 0\}. \]
∂: the path identically equal to Δ.

$k_{a_\sigma}$ is a measurable map from Ω to $W$.

Define the random time set $M(\omega)$ by

$$M(\omega) := \{t \in [0, \infty) : X_t(\omega) = a\}.$$ 

The connected components of the open set $[0, \infty) \setminus M(\omega)$ are called the *excursion intervals*.

$G(\omega)$: the collection of positive left end points of excursion intervals

By [Mai], there exists a unique $\sigma$-finite measure $P^*$ on $\Omega$ carried by $\{\sigma_a > 0\}$ and satisfying

$$E^* [1 - e^{-\sigma_a}] < \infty$$

such that

$$E_x \left[ \sum_{s \in G} Z_s \cdot \Gamma \circ \theta_s \right] = E^*(\Gamma) \cdot E_x \left[ \int_0^\infty Z_s d\ell_s \right] \quad \text{for } x \in E,$$

for any non-negative predictable process $Z$ and any non-negative random variable $\Gamma$ on $\Omega$.

($E^*$: the expectation with respect to $P^*$).

Let $Q^* = P^* \circ k_{a_\sigma}^{-1}$. $Q^*$ is a $\sigma$-finite measure on $W$ and Markovian with semigroup $\{p^*_t; t \geq 0\}$.

**Proposition 1** $Q^* = \tilde{n}$. 
§2.2 Unique determination of \( \tilde{n} \) by \( X^0, \tilde{X}^0 \)

\( m : \) a \( \sigma \)-finite Borel measure on \( E \) with \( m(\{a\}) = 0. \)

\( (u, v) = \int_E u(x)v(x)m(dx) \)

\( X = (X_t, \zeta, \mathbb{P}_x) \) and \( \tilde{X} = (\tilde{X}_t, \tilde{\zeta}, \tilde{\mathbb{P}}_x) \):
a pair of Borel right processes on \( E \) that are in weak duality with respect to \( m \); their resolvent \( G_\alpha, \tilde{G}_\alpha \) satisfy

\[
(\tilde{G}_\alpha f, g) = (f, G_\alpha g), \quad \forall f, g \in \mathcal{B}^+(E), \; \forall \alpha > 0,
\]

A point \( a \in E \) is assumed to be regular for itself and non-\( m \)-polar with respect to \( X \) and \( \tilde{X} \).

\[
\varphi(x) = \mathbb{P}_x(\sigma_a < \infty), \quad u_\alpha(x) = E_x[e^{-\alpha \sigma_a}], \quad x \in E.
\]

The corresponding functions for \( \tilde{X} \) is denoted by \( \tilde{\varphi}, \tilde{u}_\alpha \)

\( X^0, \tilde{X}^0 : \) the killed processes of \( X, \tilde{X} \) upon \( \sigma_a \).
They are in weak duality with respect to \( m_0 \).

\( \{p_t^0; t \geq 0\} \): the transition function of \( X^0 \)

For an excessive measure \( \eta \) and an excessive function \( v \) of \( X^0 \), the energy functional is defined by

\[
L^{(0)}(\eta, v) = \lim_{t \downarrow 0} \frac{1}{t} \langle \eta, v - p_t^0 v \rangle.
\]

Let \( \{\mu_t; t > 0\} \) be the \( \{p_t^0\} \)-entrance law associated with \( \tilde{n} \):

\[
\mu_t(B) = \tilde{n}(w_t \in B; t < \zeta(w)), \quad B \in \mathcal{B}(E \setminus \{a\})
\]

Then

\[
\int_W f_1(w(t_1))f_2(w(t_2)) \cdots f_n(w(t_n))n(dw)
= \mu_{t_1} f_1 P_{t_2-t_1}^0 f_2 \cdots P_{t_n-t_{n-1}}^0 f_{n-1} P_{t_n-t_{n-1}}^0 f_n.
\]
We let $\delta_0 = \mathfrak{n}(\{0\})$

**Theorem 2** (i) $\{\mu_t\}$ satisfies
\[
\tilde{\varphi} \cdot m = \int_0^{\infty} \mu_t dt.
\]

(ii) $\mathfrak{n}(W^-) = L^{(0)}(\tilde{\varphi} \cdot m, 1 - \varphi)$

(iii) It holds that
\[
L^{(0)}(\tilde{\varphi} \cdot m, 1 - \varphi) + \delta_0 = L^{(0)}(\varphi \cdot m, 1 - \tilde{\varphi}) + \hat{\delta}_0.
\]

A general theorem due to Fitzsimmons (1987):
For a transient right process with transition function $\{q_t; t \geq 0\}$, any excessive measure $\eta$ which is pure in the sense that $\eta q_t \to 0$, $t \to \infty$, can be represented by a unique $\{q_t\}$-entrance law $\{\nu_t; t > 0\}$ as
\[
\eta = \int_0^{\infty} \nu_t dt.
\]

Theorem 2 (i) means that the entrance law determining $\mathfrak{n}$ is uniquely decided by $\tilde{X}^0$ and $m$.

Theorem 2 means that the Itô point process $\mathfrak{p}$ is uniquely determined by $X^0$, $\tilde{X}^0$, $m$ up to a pair of non-negative constants $\delta_0$, $\hat{\delta}_0$ satisfying the above indentity.

Theorem 2 is a consequence of recent works by


§3. One point extensions of Brownian motions on \( \mathbb{R}^d \)

**Example 1**

\( D \subset \mathbb{R}^d \): bounded domain
\( X^0 = (X^0_t, \zeta^0, P^0_x) \): absorbing Brownian motion on \( D \)

The Dirichlet form of \( X^0 \) on \( L^2(D) \) is the Sobolev space
\( \langle \frac{1}{2} D, W^{1,2}_0(D) \rangle \), where
\[
D(u, u) = \int_D |\nabla u|^2(x)\,dx.
\]

Let
\[
\mathcal{F} = \{ w = u + c \in W^{1,2}_0(D) : c \text{ is constant} \}
\]

\[
\mathcal{E}(w, w) = \frac{1}{2} D(u, u),
\]

which is readily seen to be a regular Dirichlet form on
\( L^2(D^*; m) \)

where \( D^* = D \cup a \) is the one point compactification of \( D \)

and
\( m(dx) = 1_D(x)\,dx. \)

The associated diffusion process \( X \) on \( D^* \) extends \( X^0 \).
\( a \) is regular for itself and recurrent with respect to \( X \).

By Theorem 2 (i), the associated entrance law \( \{ \mu_t; t > 0 \} \)

equals
\[
\mu_t(B)\,dt = \int_B P^0_x(\zeta^0 \in dt)\,dx, \quad b \in \mathcal{B}(D).
\]
Example 2 (a current work with Zhen-Qing Chen)

$D \subset \mathbb{R}^d$, $d \geq 3$: unbounded uniform domain. For instance, $D$ can be an infinite cone or $\mathbb{R}^d$ itself.

Let $X = (X_t, P_x)$ be the reflecting Brownian motion on $\overline{D}$. Then $X$ is transient; it is conservative but, if $a$ denotes the point at infinity of $\overline{D}$, then

$$\lim_{t \to \infty} X_t = a \quad P_x \text{-a.s.}$$

Let $m(dx) = m(x)dx$ be a finite measure with positive density $m \in L^1(\mathbb{R}^d)$.

Let $Y = (Y_t, \zeta, P_x)$ be the time change of $X$ by its PCAF $A_t = \int_0^t m(X_s)ds$.

Then $P_x(\zeta < \infty) > 0$ and $Y_t$ approaches to $a$ as $t \to \zeta$.

**Question** How many symmetric conservative extensions does $Y$ admit?

**Answer** Only one, that can be realized as a one point extension of $Y$ to $\overline{D} \cup a$ by Itô’s ppp.

Define

$$BL(D) = \{u \in L^2_{loc}(D) : \frac{\partial u}{\partial x_i} \in L^2(D), 1 \leq i \leq d\}$$

$$W^{1,2}_e(D) = BL(D) \cap L^2(\overline{D}).$$

Then, $W^{1,2}_e(D)$ does not contain non-zero constants and

$$BL(D) = \{u + c : u \in W^{1,2}_e(D), c \text{ is constant}\}$$

$$(\frac{1}{2}D, W^{1,2}_e(D) \cap L^2(D; m))$: Dirichlet form of $Y$ on $L^2(D; m)$

$$(\frac{1}{2}D, BL(D) \cap L^2(D; m))$: its maximal Dirichlet extension on $L^2(D; m)$$