Cash flows of one dollar invested in the productive capital of the economy whose rate of return is either 5% or 1% with equal probabilities.

<table>
<thead>
<tr>
<th>Time horizon $t$</th>
<th>Cash flow $r$</th>
<th>$Ee^{rt}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$r = 5%$</td>
<td>$r = 1%$</td>
</tr>
<tr>
<td>1 year</td>
<td>1.05</td>
<td>1.01</td>
</tr>
<tr>
<td>10 years</td>
<td>1.63</td>
<td>1.10</td>
</tr>
<tr>
<td>100 years</td>
<td>131.50</td>
<td>2.70</td>
</tr>
<tr>
<td>200 years</td>
<td>17 292.58</td>
<td>7.32</td>
</tr>
</tbody>
</table>
Loan that can be obtained with a repayment of one dollar in $t$ years when the interest rate is either 5% or 1% with equal probabilities

<table>
<thead>
<tr>
<th>Time horizon $t$</th>
<th>Loan $r = 5%$</th>
<th>Loan $r = 1%$</th>
<th>$Ee^{-r}$</th>
<th>$P^r_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 year</td>
<td>0.95</td>
<td>0.99</td>
<td>0.97</td>
<td>2.96%</td>
</tr>
<tr>
<td>10 years</td>
<td>0.61</td>
<td>0.91</td>
<td>0.76</td>
<td>2.79%</td>
</tr>
<tr>
<td>100 years</td>
<td>7.60 \times 10^{-3}</td>
<td>0.37</td>
<td>0.19</td>
<td>1.68%</td>
</tr>
<tr>
<td>200 years</td>
<td>5.78 \times 10^{-5}</td>
<td>0.14</td>
<td>0.07</td>
<td>1.35%</td>
</tr>
</tbody>
</table>
First-order stochastic correlations
Definitions

- \( c_t = c_0 + x_1 + x_2 \).
- \( G(x) = \Pr[x_1 \leq x] \) and \( F(x|x_1) = \Pr[x_2 \leq x|x_1] \).
- \( (x_1, x_2) \) are "positively first-order stochastically correlated" (FSC) if \( F \) is non-increasing in \( x_1 \) for all \( x_2 \).

- Example: AR(1): \( x_2 | x_1 = \phi x_1 + \varepsilon \) with \( \phi > 0 \).
- Let \( y_2 \) denote the independent r.v. whose CDF is \( H \), with \( H(y) = \int F(y|x_1)dx_1 \).
Isolating the effect of FSC

- We compare

\[ r_t = \delta - \frac{1}{t} \ln \frac{Eu'(c_0 + x_1 + x_2)}{u'(c_0)} \]

to

\[ \tilde{r}_t = \delta - \frac{1}{t} \ln \frac{Eu'(c_0 + x_1 + y_2)}{u'(c_0)}. \]

- \( r_t \leq \tilde{r}_t \iff Eu'(c_0 + x_1 + x_2) \geq Eu'(c_0 + x_1 + y_2) \)
Two results

- **Lemma 1**: Consider any positive FSC pair $(x_1, x_2)$.

  \[ Eh(x_1, x_2) \geq Eh(x_1, y_2) \iff h \text{ is supermodular: } \frac{\partial^2 h}{\partial x_1 \partial x_2} \geq 0. \]

- **Proposition 1**: *The presence of a positive FSC correlation in changes in consumption reduces the long-term efficient discount rate if and only if the representative agent is prudent (u’ convex).*

- **Corollary**: *CRRA+FSC implies decreasing term structure*
Intuition

• The positive FSC correlation in $\Delta c$ raises the risk of the distant future compared to the i.i.d. case.

$$E(c_0 + x_1 + x_2) = E(c_0 + x_1 + y_2)$$

$$E(c_0 + x_1 + x_2)^2 \geq E(c_0 + x_1 + y_2)^2$$

• Under prudence, it is efficient to make more efforts for that distant future. This is done by reducing the long-term discount rate.
Remark: De-correlation of growth rates

- We compare

\[ r_t = \delta - \frac{1}{t} \ln \frac{E u'(c_0 e^{x_1+x_2})}{u'(c_0)} \quad \text{to} \quad \tilde{r}_t = \delta - \frac{1}{t} \ln \frac{E u'(c_0 e^{x_1+y_2})}{u'(c_0)} \]

- Proposition 2: The presence of a positive FSC correlation in changes in log consumption reduces the long-term efficient discount rate if and only if \(-u''''/u'''' > 1\).

- Notice that \(Ec_0 e^{x_1+x_2} \geq Ec_0 e^{x_1+y_2}\).
Second-order stochastic correlations
Definitions

- 
  $(x_1, x_2)$ are positive SSC if an increase in $x_1$ raises the riskiness of the $x_2|x_1$.

- Example: $x_2 \mid x_1 = \mu + \varepsilon \sqrt{x_1}$

- Lemma 2: Consider any positive SSC pair $(x_1, x_2)$.

  $Eh(x_1, x_2) \geq Eh(x_1, y_2) \iff -\frac{\partial h}{\partial x_2}$ is supermodular: $\frac{\partial^3 h}{\partial x_1 \partial x_2^2} \leq 0$. 

Results

• Proposition 3: The presence of a positive SSC correlation in changes in consumption raises the long-term efficient discount rate if and only if $u'''' < 0$.

• Intuition: Increased skewness reduces $Eu'(c_t)$ if $u'''' < 0$. 
Results

• Proposition 4: The presence of a positive SSC correlation in changes in log consumption raises the long-term efficient discount rate if and only if
  \[ f(c) = u''(c) + 3cu'''(c) + c^2u''''(c) \leq 0. \]

• Intuition: increased skewness reduces \( Eu'(c) \) if \( u'''' < 0 \).