Pricing Game Options with Call Protection: Doubly Reflected BSDEs with Call Protection and their Approximation

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Convertible bond with underlying stock $S$

- Coupons from time 0 onwards
- Terminal payoff at $\zeta = \tau \wedge \theta$

\[ 1_{\zeta = \tau \wedge S_T} + 1_{\theta < \tau} h(\theta, S_\theta) + 1_{\zeta = \tau} g(S_T) \]

- [0, $T$]-valued bond holder put time $\tau$ and bond issuer call time $\theta$
- Cancelable American claim, or game option
- Call protections preventing the issuer from calling the bond on certain random time intervals
  - Typically monitored at discrete monitoring times
  - In a possibly very path-dependent way
**Mathematical issues**

- Doubly reflected backward stochastic differential equations with an *intermittent* upper barrier, only active on random time intervals (RIBSDE)
- Related variational inequality approach (VI)
  - Highly-dimensional pricing problems (path dependence)
  - Deterministic pricing schemes ruled out by the curse of dimensionality

→ **Simulation methods**

**Contributions**

- A convergence rate for a discrete time approximation scheme by simulation to an RIBSDE
- VI approach
- Practical value of this approach on the benchmark problem of pricing by simulation highly path-dependent convertible bonds
- A demonstration of the real abilities of simulation/regression numerical schemes in high dimension (up to $d = 30$ in this work)
Outline

1. Markovian RIBSDE
   - Diffusion Set-Up with Marker Process
   - Markovian RIBSDE
   - Connection with Finance
   - Solution of the RIBSDE

2. Approximation Results
   - BSDE Approach
   - Variational Inequality Approach

3. Numerics
   - Benchmark Model
   - No Call Protection
   - Call Protection
     - Reducible Case
     - General Case
Diffusion Set-Up with Marker Process

- Diffusion with Lipschitz coefficients in $\mathbb{R}^q$
  \[ dX_t = b(t, X_t) \, dt + \sigma(t, X_t) \, dW_t \]

- Call protection monitoring times $\mathcal{Z} = \{0 = T_0 < \ldots < T_N = T\}$

- Marker process $H$ keeping track of the path-dependence, in view of ‘markovianizing’ the model

- Extended factor process $\mathcal{X} = (X, H)$ in $\mathbb{R}^q \times K$ (finite set $K$)
  - $u = u(t, x, k) = u^k(t, x)$

- $K$-valued pure jump marker process $H$ supposed to be constant except for deterministic jumps at the $T_I$s
  \[ H_{T_I} = \kappa_I(X_{T_I}, H_{T_I-}) \]

- Jump functions $\kappa^I_k$ continuous in $x$ outside $\partial \mathcal{O}$ (constant on $\mathcal{O}$ and on $^c \mathcal{O}$) for an open, ‘regular’ domain $\mathcal{O} \subseteq \mathbb{R}^q$
Call Protection

- Subset $K$ of $\mathcal{K}$
- Call forbidden/possible whenever $H_t \in K / \not\in K$
- $\mathbb{T}$-valued stopping times given as successive times of exit from and entrance to $K$, $\vartheta_0 = 0$ and then
  \[ \vartheta_{2l+1} = \inf\{t > \vartheta_{2l} ; H_t \not\in K\} \wedge T, \quad \vartheta_{2l+2} = \inf\{t > \vartheta_{2l+1} ; H_t \in K\} \wedge T \]
- Call forbidden/possible on the ‘even’/‘odd’ intervals $[\vartheta_l, \vartheta_{l+1})$
  - $H_t \in K / \not\in K$

**Starting from $H_0 = k \not\in K$ (‘Call at the beginning’)**

0 = $\vartheta_0 = \vartheta_1 < \vartheta_2 \leq \ldots \leq \vartheta_{N+1} = T$

Call possible on the first non-void time interval $[\vartheta_1 = 0 = \vartheta_0, \vartheta_2 > 0)$

**Starting from $H_0 = k \in K$ (‘Call possible at the beginning’)**

0 = $\vartheta_0 < \vartheta_1 \leq \ldots \leq \vartheta_{N+1} = T$

Call forbidden on the first non-void time interval $[\vartheta_0 = 0, \vartheta_1 > 0)$
Markovian RIBSDE

Reflected BSDE ($S$) with data

$$f(t, X_t, y, z), \xi = g(X_T), \ell(t, X_t), h(t, X_t), \vartheta$$

- ‘Standard Lipschitz and $L^2$-integrability assumptions’ (if not for $\vartheta$)
- Doubly reflected BSDE with lower barrier $L_t = \ell(t, X_t)$ and intermittent (the ‘I’ in RIBSDE) upper barriers given by, for $t \in [0, T]$

$$U_t = \sum_{l=0}^{[N/2]} 1_{[\vartheta_{2l}, \vartheta_{2l+1})} \infty + \sum_{l=1}^{[(N+1)/2]} 1_{[\vartheta_{2l-1}, \vartheta_{2l})} h(t, X_t)$$

- ‘Nominal’ upper obstacle $h(t, X_t)$ only active on the ‘odd’ random time intervals $[\vartheta_{2l-1}, \vartheta_{2l})$
- Call protection on the ‘even’ random time intervals $[\vartheta_{2l}, \vartheta_{2l+1})$
Risk-neutral pricing problems in finance

Driver coefficient function $f$ typically given as

$$f = f(t, x, y) = c(t, x) - \mu(t, x)y$$

- Dividend and interest-rate related functions $c$ and $\mu$
- Affine in $y$, does not depend on $z$
  - Historical rather than RN modeling $\rightarrow$ ‘$z$-dependent’ $f$
  - Market imperfections $\rightarrow$ nonlinear $f$
- Single-name credit risk (counterparty risk)
  - Credit-spread adjusted interest-rates $\mu$
  - Recovery-adjusted dividend-yields $c$
  - Pre-default dynamics of the factor process $X$

Terminal cost functions for instance given by

$$\ell(t, x) = \bar{P} \vee S, \quad h(t, x) = \bar{C} \vee S, \quad g(x) = \bar{N} \vee S$$

$\bar{P} \leq \bar{N} \leq \bar{C}$ Constants

$S = x_1$ first component of $x$
Highly path dependent call protection

Example (‘l out of d’)

Given a constant trigger level $\bar{S}$ and constants $l \leq d \leq N$, call possible iff $S$ has been $\geq \bar{S}$ on at least $l$ of the last $d$ monitoring times

- $\mathcal{K} = \{0, 1\}^d$, $K = \{k \in \mathcal{K}; |k| < l\}$ with $|k| = \sum_{1 \leq p \leq d} k_p$
- $\kappa^k_I(x) = (\mathbf{1}_{S \geq \bar{S}}, k_1, \ldots, k_{d-1})$
- $H_t$ vector of the indicator functions of the events $S_{T_i} \geq \bar{S}$ at the last $d$ monitoring dates preceding time $t$

Call possible iff $|H_t| \geq l \iff |H_t| \notin K$
Solution of the RIBSDE

Definition

A solution \( Y \) to \((S)\) is a triple \( Y = (Y, Z, A) \) such that:

(i) \( Y \in S^2, Z \in \mathcal{H}^2_q, A \in A^2 \)

(ii) \( Y_t = \xi + \int_t^T f(s, X_s, Y_s, Z_s)ds + A_T - A_t - \int_t^T Z_s dW_s \quad t \in [0, T] \)

(iii) \( L_t \leq Y_t \) on \([0, T]\), \( Y_t \leq U_t \) on \([0, T]\)
and \( \int_0^T (Y_t - L_t)dA_t^+ = \int_0^T (U_t - Y_t)dA_t^- = 0 \)

(iv) \( A^+ \) is continuous, and

\[ \{(\omega, t); \Delta Y \neq 0\} = \{(\omega, t); \Delta A^- \neq 0\} \subseteq \bigcup_{i=0}^{[N/2]} \{i\} \]

\( \Delta Y = \Delta A^- \) on \( \bigcup_{i=0}^{[N/2]} \{i\} \)

- \( S^2, \mathcal{H}^2_q \) and \( A^2 \) ‘usual \( L^2 \) spaces’
- \( A^\pm \) Jordan component of \( A \)
- Convention that \( 0 \times \pm \infty = 0 \) in (iii)
- Obvious extension to a random terminal time \( \theta \)
For $l$ decreasing from $N$ to 0, let us define $\mathcal{Y}^l = (Y^l, Z^l, A^l)$ on $[\vartheta_l, \vartheta_{l+1}]$ as the solution, with $A^l$ continuous, to the stopped RBSDE (for $l$ even) or R2BSDE (for $l$ odd) with data (with $Y^N_{\vartheta_{N+1}} \equiv g(X_T)$)

$$
\begin{cases}
  f(t, X_t, y, z), \ Y^l_{\vartheta_{l+1}}, \ell(t, X_t) & (l \text{ even}) \\
  f(t, X_t, y, z), \ \min(Y^l_{\vartheta_{l+1}}, h(\vartheta_{l+1}, X_{\vartheta_{l+1}})), \ell(t, X_{st}), h(t, X_t) & (l \text{ odd})
\end{cases}
$$

Let us define $\mathcal{Y} = (Y, Z, A)$ on $[0, T]$ by, for every $l = 0, \ldots, N$:

- $(Y, Z) = (Y^l, Z^l)$ on $[\vartheta_l, \vartheta_{l+1})$, and also at $\vartheta_{N+1} = T$ in case $l = N$. So in particular

$$Y_0 = \begin{cases}
  Y^0_0, & k \in K \\
  Y^1_0, & k \notin K
\end{cases}
$$

where $k$ is the initial condition of the marker process $H$.

- $dA = dA^l$ on $(\vartheta_l, \vartheta_{l+1})$, 

$$\Delta A_{\vartheta_l} = Y^l_{\vartheta_l} - \min(Y^l_{\vartheta_l}, h(\vartheta_l, X_{\vartheta_l})) = \Delta Y_{\vartheta_l} (= 0 \text{ for } l \text{ odd } \big)$$

and $\Delta A_T = \Delta Y_T = 0$.

**Proposition**

$\mathcal{Y} = (Y, Z, K)$ is the unique solution to $(S)$
Verification principle

Financial interpretation of a solution $\mathcal{V}$ to $(S)$

$\mathcal{V}_0$ Arbitrage price at time 0 for the game option with payoff functions $c, l, h, g$ and call protection $\psi$

Bilateral super-hedging price

Infimal issuer super-hedging price

$Z$ Hedging strategy
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Approximation of the Forward Process

- Time-grid $t = \{0 = t_0 < t_1 < \ldots < t_n = T\} \supseteq \mathcal{T}$
- Euler scheme approximation of $\hat{X}$
  $$\hat{X}_{t_{i+1}} = \hat{X}_{t_i} + b(t_i, \hat{X}_{t_i})(t_{i+1} - t_i) + \sigma(t_i, \hat{X}_{t_i})(W_{t_{i+1}} - W_{t_i})$$
- Approximation of the marker process $H$
  $$\hat{H}_{T_i} = \kappa_I(\hat{X}_{T_i}, \hat{H}_{T_i-})$$
Approximation $\hat{\vartheta}$ of $\vartheta$ obtained by using $\hat{\mathcal{X}} = (\hat{X}, \hat{H})$ instead of $\mathcal{X}$ in the definition of $\vartheta$

**Proposition (Assuming $\sigma$ non-degenerate and ‘some regularity of $\sigma$ and $b$ around $\partial \mathcal{O}$)**

For every $l \leq N + 1$

$$\mathbb{E}\left[|\vartheta_l - \hat{\vartheta}_l|\right] \leq C_\varepsilon |t|^{\frac{1}{2} - \varepsilon}$$

$$|t| = \max_{i \leq n-1} (t_{i+1} - t_i)$$
Approximation of the RIBSDE

- Projection operator \( \hat{P} \) defined by

\[
\hat{P}(t, x, y) = y + [\ell(t, x) - y]^+ - [y - h(t, x)]^+ \sum_{l=1}^{[(N+1)/2]} 1_{\{\vartheta_{2l-1} \leq t \leq \vartheta_{2l}\}}
\]

- Reflection operating only on a subset \( \tau \) of \( t \) in the approximation scheme for \( Y \)

\[
\tau = \{0 = r_0 < r_1 < \cdots < r_\nu = T\} \text{ with } \mathcal{T} \subseteq \tau \subseteq \mathcal{T}
\]
Components $Y$ and $Z$ of a solution $\mathcal{Y} = (Y, Z, A)$ to $(S)$ approximated by a triplet of processes $(\hat{Y}, \tilde{Y}, \tilde{Z})$ defined on $t$

Terminal condition

$$\hat{Y}_T = \tilde{Y}_T = g(\hat{X}_T)$$

and then for $i$ decreasing from $n - 1$ to 0

$$\begin{align*}
\tilde{Z}_{t_i} &= \mathbb{E}\left[ \hat{Y}_{t_{i+1}} \left( \frac{W_{t_{i+1}} - W_{t_i}}{t_{i+1} - t_i} \right) | \mathcal{F}_{t_i} \right] \\
\tilde{Y}_{t_i} &= \mathbb{E}\left[ \hat{Y}_{t_{i+1}} | \mathcal{F}_{t_i} \right] + (t_{i+1} - t_i)f(t_i, \hat{X}_{t_i}, \tilde{Y}_{t_i}, \tilde{Z}_{t_i}) \\
\hat{Y}_{t_i} &= \tilde{Y}_{t_i} 1_{\{t_i \notin \tau\}} + \mathcal{P}(t_i, \hat{X}_{t_i}, \tilde{Y}_{t_i}) 1_{\{t_i \in \tau\}}
\end{align*}$$

Conditional expectations well defined at each step

- Square integrable processes
Theorem (No call or no call protection, Chassagneux 08)

\[
\max_{i \leq n-1} \sup_{t \in [t_i, t_{i+1})} \mathbb{E}[|Y_t - \tilde{Y}_{t_i}|^2] + \max_{i \leq n-1} \sup_{t \in [t_i, t_{i+1})} \mathbb{E}[|Y_{t-} - \tilde{Y}_{t_i}|^2] \leq C|t|^\alpha
\]

\[\alpha \frac{1}{3} \text{ in case of Lipschitz barriers or } \frac{1}{2} \text{ in case of semi-convex barriers}\]

Theorem (Call protection, this work, assuming that } f \text{ does not depend on } z)

\[
\max_{i \leq n-1} \sup_{t \in [t_i, t_{i+1})} \mathbb{E}[|Y_t - \tilde{Y}_{t_i}|^2] + \max_{i \leq n-1} \sup_{t \in [t_i, t_{i+1})} \mathbb{E}[|Y_{t-} - \tilde{Y}_{t_i}|^2] \leq C_\varepsilon|t|^{\alpha - \varepsilon}
\]

\[\alpha \frac{1}{6} \text{ in case of Lipschitz barriers or } \frac{1}{4} \text{ in case of semi-convex barriers}\]
• Proof of the theorem based on a suitable concept of (time-continuous) discretely reflected BSDEs
• Possible extension to the case where $f$ depends on $z$
• Representations of $\tilde{Y}$ and $\tilde{Z}$ using approximated optimal policies
  • Cf. ‘MC Backward versus Forward’ in the numerical part
Comparing the simulation results them with those of an alternative, deterministic numerical scheme

Deterministic scheme for (S) based on an analytic characterization of (S)

Let $\mathcal{E} = [0, T] \times \mathbb{R}^q \times \mathcal{K}$ and for $i = 1, \ldots, N$

$$\mathcal{E}_i = \mathcal{E} \cap ([T_{i-1}, T_i] \times \mathbb{R}^q \times \mathcal{K}) \ , \ \mathcal{E}^*_i = \mathcal{E} \cap ([T_{i-1}, T_i) \times \mathbb{R}^q \times \mathcal{K})$$

The $\mathcal{E}^*_i$s and $\{ T \} \times \mathbb{R}^q \times \mathcal{K}$ partition $\mathcal{E}$
# Cauchy cascade

## Definition

1. **Cauchy cascade** \((g, \nu)\) on \(\mathcal{E}\)
   - **Terminal condition** \(g\) at \(T\)
   - **Sequence** \(\nu = (u_1)_{1 \leq i \leq N}\) of functions \(u_1\)s on the \(\mathcal{E}_i\)s
   - **Jump condition** for \(x \notin \partial \mathcal{O}\) (with \(u_{N+1} \equiv g\)):

\[
    u^k_i(T_i, x) = \begin{cases} 
    \min(u_{i+1}(T_i, x, \kappa^k_i(x)), h(T_i, x)) & \text{if } k \notin K \text{ and } \kappa^k_i(x) \in K, \\
    u_{i+1}(T_i, x, \kappa^k_i(x)) & \text{else}
    \end{cases}
\]

2. **Continuous Cauchy cascade**
   - Cauchy cascade with continuous ingredients \(g\) at \(T\) and \(u_1\)s on the \(\mathcal{E}_i\)s, except maybe for discontinuities of the \(u^k_i\)s on \(\mathcal{Z} \times \partial \mathcal{O}\)
   (note \(\not\equiv\) not ‘lag’ on \(\mathcal{Z} \times \partial \mathcal{O} \times K\))

3. **Function on \(\mathcal{E}\) defined by a Cauchy cascade**
   - Concatenation on the \(\mathcal{E}_i^*\)s of the \(u_1\)s + terminal condition \(g\) at \(T\)
Cascade Characterization of $Y$

Proposition

$Y_t = u(t, X_t)$, $t \in [0, T]$, for a deterministic pricing function $u$, defined by a continuous Cauchy cascade $(g, \nu = (u_I)_{1 \leq I \leq N})$ on $\mathcal{E}$

Analytic characterization of $u$?

Generator of $X$

$G \phi(t, x) = \partial_t \phi(t, x) + \partial \phi(t, x)b(t, x) + \frac{1}{2}\text{Tr}[a(t, x)\mathcal{H}\phi(t, x)]$

$a(t, x) \sigma(t, x)\sigma(t, x)^T$

$\partial \phi$, $\mathcal{H}\phi$ Row-gradient and Hessian of $\phi$ with respect to $x$
Cauchy cascade ($\mathcal{VI}$)

For $l$ decreasing from $N$ to 1,

- At $t = T_l$ for every $k \in \mathcal{K}$ and $x \in \mathbb{R}^q$

\[
    u^k_l(T_l, x) = \begin{cases} 
    \min(u_{l+1}(T_l, x, \kappa^k_l(x)), h(T_l, x)), & \text{if } k \notin \mathcal{K} \text{ and } \kappa^k_l(x) \in \mathcal{K} \\
    u_{l+1}(T_l, x, \kappa^k_l(x)), & \text{else}
    \end{cases}
\]

with $u_{N+1} \equiv g$

- On the time interval $[T_{l-1}, T_l)$ for every $k \in \mathcal{K}$,

\[
\begin{align*}
    &\min \left( - \mathcal{G} u^k_l - f u^k_l, u^k_l - \ell \right) = 0, \quad k \in \mathcal{K} \\
    &\max \left( \min \left( - \mathcal{G} u^k_l - f u^k_l, u^k_l - \ell \right), u^k_l - h \right) = 0, \quad k \notin \mathcal{K}
\end{align*}
\]

with for any function $\phi = \phi(t, x)$

\[
f^\phi = f^\phi(t, x) = f(t, x, \phi(t, x))
\]
Technical difficulty due to the potential discontinuity in $x$ of the functions $u^k$ on $\partial \Omega$

- Characterizing $\nu$ in terms of a suitable notion of discontinuous viscosity solution of $(\mathcal{VI})$?
- Convergence results? for standard deterministic (like finite differences) approximation schemes to $u$

Curse of dimensionality

- $(\mathcal{VI}) = \text{Card}(\mathcal{K})$ equations in the $u^k$'s
- $\sim (q + d)$ – dimensional pricing problem with $d = \log(\text{Card}(\mathcal{K}))$
- Simulation schemes the only viable alternative for $d$ greater than few units
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Benchmark Model

Local drift and volatility pre-default model for a stock $X = S$

\[
\frac{dS_t}{S_t} = b(t, S_t) dt + \sigma(t, S_t) dW_t
\]

\[b(t, S) = r(t) - q(t) + \eta \gamma(t, S), \quad \mu(t, S) = r(t) + \gamma(t, S)\]

\[\gamma(t, S) = \gamma_0(S_0/S)^\alpha, \quad \sigma(t, S) = \sigma\]

- $r(t)$: Riskless short interest rate
- $q(t)$: Dividend yield
- $\gamma(t, S)$: Local default intensity

0 $\leq \eta \leq 100\%$ of the firm issuing the bond ($\gamma_0, \alpha \geq 0$)

\[\beta_t = e^{-\int_0^t \mu(s, S_s) ds}\] Risk-neutral credit-risk adjusted discount factor

\[\mu(t, S) = r(t) + \gamma(t, S)\] Credit-risk adjusted interest rate

Coupon rate function

\[c(t, S) = \bar{c}(t) + \gamma(t, S) ((1 - \eta)S \vee \bar{R})\]

- $\bar{c}$: Nominal coupon rate function
- $\bar{R}$: Nominal recovery on the bond upon default
General Conditions for the Numerical Experiments

General Data

<table>
<thead>
<tr>
<th>P</th>
<th>N</th>
<th>C</th>
<th>μ</th>
<th>σ</th>
<th>r</th>
<th>q</th>
<th>γ₀</th>
<th>α</th>
<th>m</th>
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<td>0</td>
<td>100</td>
<td>103</td>
<td>1</td>
<td>0.2</td>
<td>0.05</td>
<td>0</td>
<td>0.02</td>
<td>1.2</td>
<td>10⁴</td>
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</table>

\( m \) number of Monte Carlo simulations (trajectories) which are used in the stochastic pricing schemes

Time-step \( t_{i+1} - t_i = h \)

- six hours (four time steps per day) in the case of simulation methods
- one day in the case of deterministic schemes

Space-steps in the \( S \) variable

\( S^{i+1} - S^i = 0.5 \) in the case of the (fully implicit) deterministic schemes

Cells of diameter one (segments of length one) in the case of simulation/regression methods involving a method of cells in \( S \)
No Call Protection

\( \vartheta_1 = 0, \vartheta_2 = T \)

Simulated mesh \((S^j_i)_{0 \leq i \leq n} \) → Estimate \((u^j_i) = u(t_i, S^j_i)_{1 \leq j \leq m} \)

\( u_n = g \), then for \( i = n - 1 \ldots 0 \), for \( j = 1 \ldots m \),

\[
    u^j_i = \min \left( h_i(S^j_i), \max \left( \ell_i(S^j_i), e^{-\mu_i h} \mathbb{E}^j_i(u_{i+1} + hc_{i+1}) \right) \right)
\]

\( \mathbb{E}^j_i(u_{i+1} + hc_{i+1}) \) Conditional expectation given \( t = t_i, S_i = S^j_i \)

- Computed by **non-linear regression** of \((u_{i+1} + hc_{i+1})_{1 \leq j \leq m} \) against \((S_i)_{1 \leq j \leq m} \), using a global parametric regression basis 1, \( S \), \( S^2 \) in \( S \)

Regression estimate of the delta

\[
    \delta^j_i = \frac{\mathbb{E}^j_i \{ u_{i+1}(W_{i+1} - W_i) \}}{\sigma_i(S^j_i) S^j_i h}
\]

Alternative **MC forward** estimates of price and delta at time 0
MC Backward vs Forward

Maturity $T = 125$ days, Nominal coupon rate $\bar{c} = 0$

MC Fd less volatile than MC Bd (Deviations over 50 trials, $S_0 = 100.55$)

<table>
<thead>
<tr>
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<th>Value VI</th>
<th>Dev MC Bd</th>
<th>Dev MC Fd</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price</td>
<td>102.049</td>
<td>0.821</td>
<td>0.010</td>
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<tr>
<td>Delta</td>
<td>0.416</td>
<td>0.071</td>
<td>0.019</td>
</tr>
</tbody>
</table>

MC Fd more accurate than MC Bd (%Err=1 $\leftrightarrow$ relative difference of 1% between MC and VI)

<table>
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<tr>
<th>$S_0$</th>
<th>VI Price</th>
<th>%Err Bd</th>
<th>%Err Fd</th>
<th>VI delta</th>
<th>%Err Bd</th>
<th>%Err Fd</th>
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<td>1.90</td>
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<tr>
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<td>1.99</td>
<td>0.01</td>
<td>0.416</td>
<td>2.77</td>
<td>0.67</td>
</tr>
<tr>
<td>101.55</td>
<td>102.479</td>
<td>1.65</td>
<td>0.07</td>
<td>0.435</td>
<td>3.97</td>
<td>3.47</td>
</tr>
</tbody>
</table>

In the sequel always use MC forward estimates
Non-decreasing sequence of $[0, T]$-valued stopping times $\vartheta = (\vartheta_l)_{l \geq 0}$

Effective call payoff process

$$U_t = \Omega_t^\infty + \Omega_t h(t, X_t) = U(t, S_t, H_t)$$

$$\Omega_t = 1\{t \text{ odd}\} = 1\{H_t \notin K\} = \Omega(t, S_t, H_t) \text{ with } \vartheta_{l_t} \leq t < \vartheta_{l_t+1}$$

Simulated mesh $(S^i_j, H^i_j)_{0 \leq i \leq n}$ → Estimate $(u^i_j) = u(t_i, S^i_j, H^i_j)_{1 \leq j \leq m}$

$u_n = g$, then for $i = n - 1 \ldots 0$, for $j = 1 \ldots m$

$$u^i_j = \min \left( U_i \left( S^i_j, H^i_j \right), \max \left( \ell_i \left( S^i_j \right), e^{-rh} \mathbb{E}^i_j (u_{i+1} + hc_{i+1}) \right) \right)$$

min plays no role outside the support of $\Omega$, where $U_i (S, H)$ is equal to $+\infty$
$\mathbb{E}_i^j(u_{i+1} + h c_{i+1})$ Conditional expectation given $t = t_i, S_i = S_i^j, H_i = H_i^j$

- computed by non-linear regression of $(u_{i+1} + h c_{i+1})_{1 \leq j \leq m}$ against $(S_i, H_i)_{1 \leq j \leq m}$, using for example a method of cells in $(S, H)$

More Data

<table>
<thead>
<tr>
<th>$\Omega(t, S, H)$</th>
<th>$\bar{S}$</th>
<th>$T$</th>
<th>$\bar{c}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>'l out of d'</td>
<td>103</td>
<td>180 days</td>
<td>1.2/month</td>
</tr>
</tbody>
</table>
In case $l = d$ one can reduce the problem to two space dimensions instead of $d + 1$

- $S$ and the number $N$ of consecutive monitoring dates $T_i$s with $S_{T_i} \geq \bar{S}$ from time $t$ backwards (capped at $l$)

Two simulation schemes

- $MC_d$ a method of cells in $(S, H)$
- $MC^1$ a method of cells in $(S, N)$

$MC_d$ more accurate than $MC^1$ ($S_0 = 100$)

<table>
<thead>
<tr>
<th>$l$</th>
<th>$1$</th>
<th>$5$</th>
<th>$10$</th>
<th>$20$</th>
<th>$30$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$VI^1$ price</td>
<td>103.91</td>
<td>105.10</td>
<td>106.03</td>
<td>107.22</td>
<td>108.01</td>
</tr>
<tr>
<td>$MC^1$ %Err</td>
<td>0.04</td>
<td>0.16</td>
<td>0.47</td>
<td>0.88</td>
<td>1.34</td>
</tr>
<tr>
<td>$MC_d$ %Err</td>
<td>0.04</td>
<td>0.15</td>
<td>0.03</td>
<td>0.04</td>
<td>0.24</td>
</tr>
</tbody>
</table>
General Case

<table>
<thead>
<tr>
<th></th>
<th>$d$</th>
<th>1</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>VI$_d$</strong></td>
<td></td>
<td>332s</td>
<td>5332s</td>
<td>44h</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>MC$_d$</strong></td>
<td></td>
<td>154s</td>
<td>212s</td>
<td>313s</td>
<td>474s</td>
<td>628s</td>
</tr>
<tr>
<td><strong>Rel Err</strong></td>
<td></td>
<td>range 1 bp—1%</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Will use two methods for the computation of the conditional expectations in MC$_d$:

- MC$_d$  a method of cells in $(S, H)$,
- MC$_d^\#$ a method of cells in $(S, |H|^\#)$
Approximate $\text{MC}_d^\#$ Algorithm

$|H|^\#$ number of ones in $H$ starting from the $(l - |H|)^{th}$ zero

Example ($d = 10$, $l = 8$)

- $H = (1, 1, 1, 1, 0, 1, 1, 1, 0, 0)$
  
  $l - |H| = 8 - 7 = 1$, $|H|^\# = 3$ (number of ones on the right of the first zero, in bold in $H$),

- $H = (1, 1, 1, 0, 1, 1, 1, 0, 0, 0)$ $l - |H| = 8 - 6 = 2$, $|H|^\# = 0$ (number of ones on the right of the second zero, in bold in $H$)

Rationale

Entries of $H$ preceding its $(l - |H|)^{th}$ irrelevant to the price

- Necessarily superseded by new ones before the bond may become callable

- Approximate algorithm based on the ‘good regressor’ $|H|^\#$ for estimating highly path-dependent conditional expectations
### Benchmark Model

#### No Call Protection

**MC\(_d\) good, MC\(^\#_d\) ‘rather good’ (\(d = 5, S_0 = 100\))**

<table>
<thead>
<tr>
<th>(l)</th>
<th>2</th>
<th>3</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>VI(_d) price</td>
<td>104.07</td>
<td>104.43</td>
<td>105.10</td>
</tr>
<tr>
<td>MC(_d) %Err</td>
<td>0.21</td>
<td>0.15</td>
<td>0.15</td>
</tr>
<tr>
<td>MC(^#_d) %Err</td>
<td>0.19</td>
<td>0.23</td>
<td>0.18</td>
</tr>
</tbody>
</table>

**MC\(_d\) good, MC\(^\#_d\) ‘OK’ (\(d = 10, S_0 = 100\))**

<table>
<thead>
<tr>
<th>(l)</th>
<th>2</th>
<th>5</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>VI(_d) price</td>
<td>104.27</td>
<td>104.87</td>
<td>106.03</td>
</tr>
<tr>
<td>MC(_d) %Err</td>
<td>0.01</td>
<td>0.15</td>
<td>0.03</td>
</tr>
<tr>
<td>MC(^#_d) %Err</td>
<td>0.04</td>
<td>0.26</td>
<td>0.38</td>
</tr>
</tbody>
</table>
Deviation over 50 trials and relative difference \((d = 30, S_0 = 102.55)\)

<table>
<thead>
<tr>
<th>(l)</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dev MC(_d)</td>
<td>0.056</td>
<td>0.061</td>
<td>0.086</td>
<td>0.152</td>
</tr>
<tr>
<td>Dev MC(^#)_d</td>
<td>0.060</td>
<td>0.069</td>
<td>0.092</td>
<td>0.175</td>
</tr>
<tr>
<td>% Err</td>
<td>0.09</td>
<td>0.24</td>
<td>0.72</td>
<td>1.06</td>
</tr>
</tbody>
</table>

‘Good regressor’ algorithm MC\(^\#\)_d rather accurate in practice

Ability to work with a ‘good’ (as opposed to exact), low-dimensional regressor

- An interesting feature of simulation as opposed to deterministic numerical schemes
Proof of the Cascade Characterization of $\mathcal{V}$

Non-linear Regressions
Markov Families Embedding

- Everything implicitly parameterized by the initial condition \((t = 0, x, k)\) of \(X\)
- Superscript \(t\) in reference to an initial condition \((t, x, k)\) of \(X\) (with in particular \(t \in [0, T]\) rather than \(t = 0\) implicitly above)
- \(Y^t = Y^{t, l}\) on \([\vartheta^t_i, \vartheta^t_{i+1})\), and in particular

\[
Y^t = \begin{cases} 
Y^0_t, & k \in K \\
Y^1_t, & k \notin K 
\end{cases}
\]

\[
\begin{align*}
Y^0_{\vartheta^t_1} &= Y^1_{\vartheta^t_1} \\
Y^1_{\vartheta^t_2} &= \min \left( Y^2_{\vartheta^t_2}, h(\vartheta^t_2, X^t_{\vartheta^t_2}) \right)
\end{align*}
\]
Markovianity of $Y$

Standard semi-group properties of $X$ and $Y$ (SDEs uniqueness results) yield, for every $I = 1, \ldots, N$ and $T_{I-1} \leq t < r < T_I$,

$$Y_t^r = u_I(r, X_t^r), \text{ } \mathbb{Q}-a.s.$$  

for a deterministic function $u_I$ on $\mathcal{E}_I^*$. In particular,

$$Y_t^t = u^k(t, x), \text{ for any } (t, x, k) \in \mathcal{E}$$

where $u$ is the function defined on $\mathcal{E}$ by the concatenation of the $u_I$'s and the terminal condition $g$ at $T$. 

Chassagneux, Crépey, Rahal

RIBSDEs
Continuity of the $u^k$'s outside $\mathcal{T} \times \partial \mathcal{O}$

Case $t \notin \mathcal{T}$

- $u^k(t, x)$ identified ‘in the vicinity of $(t, x)$’ to
  - $Y^{0,t}_t$ if $k \in K$ (no call at the beginning)
  - $Y^{1,t}_t$ if $k \notin K$ (call at the beginning)
- + stability estimates on the $Y^{l,t}_t$'s $\rightarrow u^k$ continuous at $(t, x)$
- Also shows that $u^k$ is ‘cad’ at every $(t = T_i, x)$

Remains to show that

- the $u_I$'s can be extended by continuity over the $\mathcal{E}_I$'s, except maybe at the boundary points $(T_I, x \in \partial \mathcal{O}, k)$
- the jump condition is satisfied
Given $\mathcal{E}_i^* \ni (t_n, x_n, k) \to (t = T_i, x, k)$ with $x \notin \partial \mathcal{O}$

- Needs to show that $u_i^k(t_n, x_n) = u^k(t_n, x_n) \to u_i^k(T_i, x)$, with $u_i^k(T_i, x)$ given by the jump condition

- Note $\vartheta$ ‘cadlag’ at $(t = T_i, x)$
  - $\vartheta^{t_n} \to \tilde{\vartheta}^t$ as $n \to \infty$, where $\tilde{\vartheta}^t$ ‘jumps at $t = \tilde{\vartheta}^t$’

- Intuition ‘$\tilde{\mathcal{Y}} = \mathcal{Y} \circ \kappa$’, and so
  ‘$\lim_{n\to\infty} u^k(t_n, x_n) = \tilde{u}(t, x, k) = u(t, x, \kappa_i^k(x))$’

- Obviously misses some point since, in case for instance $k \notin K$ and $\kappa_i^k(x) \in K$, one ‘clearly’ has ‘$\tilde{u}(t, x, k) \leq h(t, x)$’, which is not necessarily satisfied by $u(t, x, \kappa_i^k(x))$

In fact in case $k \notin K$ and $\kappa_i^k(x) \in K$ one has consistently with the jump condition that $\tilde{u}(t, x, k) = \min\left(u(t, x, \kappa_i^k(x)), h(t, x)\right)$, as we now prove. The other three cases can be proven similarly.
Denoting $\tilde{u}^i(s, y) = \min \left( u(s, y, \kappa^i_i(y)), h(s, y) \right)$ and 
$\hat{u}^i(s, y) = \min \left( u^i(s, y), h(s, y) \right)$

$$|	ilde{u}^k(t, x) - u^k(t_n, x_n)|^2 = |	ilde{u}^k(t, x) - Y_{t_n}^{1,t_n}|^2 \leq 2\mathbb{E}|	ilde{u}^k(t, x) - \hat{u}(t, \mathcal{X}^n_t)|^2 + 2\mathbb{E}(\hat{u}(t, \mathcal{X}^n_t) - Y_{t_n}^{1,t_n})^2$$

- $(t, \mathcal{X}^n_t) \in \mathcal{E}_{i+1}^*$ and ‘close to $(t, x, \kappa^k_i(x))$’ → first term goes to 0 by continuity of $u$ already established over $\mathcal{E}_{i+1}^*$
- $u(t, \mathcal{X}^n_t) = Y_{t_n}^t$ and $t \sim \mathcal{Y}_2^n$ so

$\hat{u}(t, \mathcal{X}^n_t) = \min \left( u(\mathcal{Y}_2^n, \mathcal{X}^n_{\mathcal{Y}_2^n}), h(\mathcal{Y}_2^n, \mathcal{X}^n_{\mathcal{Y}_2^n}) \right)$

$= \min \left( Y_{\mathcal{Y}_2^n}^{2,t_n}, h(\mathcal{Y}_2^n, \mathcal{X}^n_{\mathcal{Y}_2^n}) \right) = Y_{\mathcal{Y}_2^n}^{1,t_n} = Y_{t_n}^{1,t_n}$

→ second term goes to zero by $(\mathcal{E}^n_t)$ and convergence of $\mathcal{Y}_{t_n}$ (to $\hat{\mathcal{Y}}_t$)
Proof of the Cascade Characterization of $\mathcal{Y}$

Non-linear Regressions
Non-linear simulation/regression approaches for computing by regression functions (conditional expectations)

\[ x \mapsto \rho(x) = \mathbb{E}(\xi|X = x) \]

\( \xi, X \) Real- and \( \mathbb{R}^q \)-valued square integrable random variables

Pairs \( (X_j, \xi_j)_{1 \leq j \leq m} \) simulated independently according to the law of \( (X, \xi) \) → Estimate the conditional expectation \( \mathbb{E}(\xi|X) \)

Linear regression of the \( \xi_j \)'s against the \( (\varphi^l(X^j))_{1 \leq j \leq m} \), where \( (\varphi^l)_{l \in \mathbb{N}} \), is a well chosen ‘basis’ of functions from \( \mathbb{R}^q \) to \( \mathbb{R} \)

Regression basis

- parametric vs non-parametric
- global vs local
Typically

- parametric and global
  - few monomials parameterized by their coefficients
- or non-parametric and local
  - indicator functions of the cells of a grid of hyperrectangles partitioning the state space

Preferred

- Global basis in case of a ‘regular’ regression function $\rho(x)$
  - Case where a good guess is available as for the shape, used to define the regression basis, of $\rho$
- Local basis otherwise
  - Often simpler and more robust in terms of implementation