Liquidity Risk Measures

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Liquidity

- **Illiquidity poses significant risk for investors**
  - Short-term obligations can force asset holders to liquidate assets
  - In illiquid markets, prices of fire sales may be suboptimal

- **Liquidity-Adjustments**
  - The value of portfolios should be adjusted for these adverse scenarios
  - The liquidity-risk should be quantified

- **Important model ingredients**
  - Supply-demand curves
  - Short-term obligations
Outline

(i) Liquidity Constraints
   • Supply-Demand Curves
   • Liquidity and Portfolio Constraints

(ii) Liquidity Adjustments
   • Portfolio Value
   • Risk Measures

(iii) Numerical Case Studies
Liquidity Constraints
Supply-Demand Curves

• Marginal supply-demand curve (MSDC)
  \( m : \mathbb{R}_* \to \mathbb{R} \) decreasing function
  – \( y > 0 \):
    Sell \( y \) shares for \( \int_0^y m(x)dx \)
  – \( y < 0 \):
    Buy \(|y|\) shares for \( \int_y^0 m(x)dx \), i.e. “receive” \( \int_y^0 m(x)dx \)

The convex cone of all MSDC’s is denoted by \( \mathcal{M} \).

• The numbers \( m^+ := m(0+) \) and \( m^- := m(0-) \) are called the best bid and best ask, respectively.

• The difference \( \Delta m : m^- - m^+ \geq 0 \) is called the bid-ask spread.
Supply-Demand Curves (cont.)

- **Proceeds of transaction:** \( P(s) = \int_0^s m(x)dx \)
- **Supply-demand curves:**
  \[
  S(x) := \frac{P(x)}{x}, \quad x \in \mathbb{R}_*
  \]
  - \( x > 0 \): average unit prices for sale
  - \( x > 0 \): average unit prices for purchase
Examples

Linear Supply-Demand Curve

- Supply-demand curve: \( S(x) = a - b \cdot x \) for given \( a, b > 0 \)
- Proceeds: \( P(x) = x \cdot S(x) = x \cdot (a - b \cdot x) \)
- Marginal supply-demand curve: \( m(x) = a - 2b \cdot x \)

Remark

Marginal prices become negative, if seller sells more than \( a/2b \) units. This case will, however, not occur, since it is never optimal for the seller to sell at these prices.
Examples (cont.)

Exponential Supply-Demand Curve

- Marginal supply-demand curve: \( m(x) = a \cdot e^{-bx} \) for \( a, b > 0 \)
- Proceeds: \( S(x) = \frac{a}{bx} \cdot (1 - e^{-bx}) \)
- Supply-demand curve: \( P(x) = \frac{a}{b} \cdot (1 - e^{-bx}) \)

Polynomial Supply-Demand Curve

- Marginal supply-demand curve: \( m(x) = \begin{cases} a(b - x)^\gamma, & x < b, \\ 0, & x \geq b. \end{cases} \)
  for \( a, b > 0 \) and \( \gamma > 1 \)
- Proceeds: \( P(x) = \frac{a}{\gamma+1}(b^{\gamma+1} - (b - x)^{\gamma+1}) \)
- Supply-demand curve: \( S(x) = \frac{a}{(\gamma+1)x} \cdot (b^{\gamma+1} - (b - x)^{\gamma+1}) \)
Markets and Portfolios

- **Spot market** of assets:
  \[ \vec{m} = (m_0, m_1, \ldots, m_N) \in \mathcal{M}^{N+1}. \]
  We will always assume that asset 0 corresponds to cash and set \( m_0 = 1. \)

- **Portfolio** in a spot market of \( N \) risk assets:
  \[ \vec{\xi} = (\xi_0, \xi_1, \ldots, \xi_N) = (\xi_0, \xi) \in \mathbb{R}^{N+1} \]
  "Number of assets at time 0"
Liquidity Risk Measures

Liquidation versus Mark-to-Market Value

- Liquidation value

\[ L(\bar{\xi}, \bar{m}) = \sum_{i=0}^{N} \int_{0}^{\xi_i} m_i(x)dx = \xi_0 + \sum_{i=1}^{N} \int_{0}^{\xi_i} m_i(x)dx. \]

- Maximal mark-to market value

\[ U(\bar{\xi}, \bar{m}) = \xi_0 + \sum_{i=1}^{N} \tilde{m}_i(\xi_i) \cdot \xi_i = L(\bar{\xi}, \bar{m}) - \sum_{i=1}^{N} \int_{0}^{\xi_i} \hat{m}_i(x)dx, \]

where \( \tilde{m}_i(\xi_i) = \begin{cases} 
  m_i^+, & \text{if } \xi_i \geq 0, \\
  m_i^-, & \text{if } \xi_i < 0.
\end{cases} \]
Attainable Portfolios

- **Portfolio at time 0**: $\bar{\xi} \in \mathbb{R}^{N+1}$

- Any portfolio which is **attainable from** $\bar{\xi}$ has the form:

$$
\left( \xi_0 + \sum_{i=1}^{N} \int_{0}^{\eta_i} m_i(x) \, dx, \xi - \eta \right) \quad (\eta \in \mathbb{R}^N)
$$

We denote the set of **attainable portfolios** by $\mathcal{A}(\bar{\xi}, \bar{m})$.

**Remark**

The maximal mark-to-market values of the attainable portfolios will typically be an interval that is bounded above by the mark-to-market of the original portfolio.
Liquidity Adjustments
Liquidity Constraints

• **Short-term cash flows**

  Continuous function \( \phi : \mathbb{R}^N \rightarrow \mathbb{R} \cup \{-\infty\} \) such that \( \phi(0^N_N) = 0 \).

  We will usually assume that \( \phi \) is concave and non positive.

  Notation: \( \phi \in \Phi \)

• **Set of liquid portfolios**

  Letting \( \phi \in \Phi \) and \( a \in \mathbb{R} \), the set of liquid portfolios that are attainable from \( \bar{\xi} \) are defined as

  \[
  \mathcal{L}(\bar{\xi}) = \mathcal{L}(\bar{\xi}, \bar{m}, \phi, c) = \{ \bar{\eta} \in A(\bar{\xi}, \bar{m}) : \eta_0 + \phi(\eta) \geq a \}.
  \]
Liquidity Constraints (cont.)

Examples

- **Proportional margin constraints:**
  the obligations are proportional to the number of assets on which the investor is short, i.e.
  \[
  \phi(\xi) = - \sum_{i=1}^{N} \alpha_i \cdot \xi_i, \quad \alpha_i \geq 0, \quad i = 1, \ldots, N
  \]

- **Quadratic margin constraints:** the obligations are quadratic in the number of assets on which the investor is short, i.e.
  \[
  \phi(\xi) = - \sum_{i=1}^{N} \alpha_i \cdot (\xi_i^{-})^2, \quad \alpha_i \geq 0, \quad i = 1, \ldots, N
  \]
Portfolio Constraints

Let \( \mathcal{K} \subseteq \mathbb{R}^N \) be non-empty, closed, convex set.

Requirement

- \( \eta \in \mathcal{K} \) for any admissible portfolio \( \bar{\eta} = (\eta_0, \eta) \) at the end of the time horizon, \( t = 1 \)

- We suppose that \( 0_N \in \mathcal{K} \), i.e. holding cash only is acceptable, as long as the borrowing constraint \( \eta_0 \geq a \) is satisfied.

We will always assume that the portfolio constraint can be expressed in terms of \( r \) convex functions \( \psi_1, \ldots, \psi_r : \mathbb{R}^N \rightarrow \mathbb{R} \), i.e.

\[
\eta \in \mathcal{K} \iff \psi_1(\eta) \leq 0, \ldots, \psi_r(\eta) \leq 0.
\]
Portfolio Value

Attainable portfolios

\[ \mathcal{L}(\bar{\xi}) = \mathcal{L}(\bar{\xi}, \bar{m}, \phi, c) = \{ \bar{\eta} \in \mathcal{A}(\bar{\xi}, \bar{m}) : \eta_0 + \phi(\eta) \geq a \} \]

Value of a portfolio

\[ V(\bar{\xi}, \bar{m}, \phi, a, \mathcal{K}) = \sup \{ U(\bar{\eta}, \bar{m}) : \bar{\eta} \in \mathcal{L}(\bar{\xi}, \bar{m}, \phi, a) \cap \mathcal{K} \} . \]

(Acerbi & Scandolo, 2008; Anderson, Liese & W., 2009)
Liquidity Risk Measures

- Assume that the MSDC $m$ and the ICF $\phi$ are random.
- Let $A$ be the acceptance set of a risk measure $\rho$.

Liquidity-based risk measure (Anderson, Liese & W., 2009)

- The risk of a portfolio $\bar{\xi}$ is defined as
  \[ \rho^V(\bar{\xi}) = \inf \{ k : V(k + \bar{\xi}) \in A \} \]

- The mapping $\rho^V : \mathbb{R}^{N+1} \rightarrow \mathbb{R}$ is a convex risk measure.
- $\rho^V(\bar{\xi})$ is equal to the unique solution $k \in \mathbb{R}$ of the equation
  \[ 0 = \rho(V(\bar{\xi} + k)) \]
- In the case of UBSR this amounts to solving the equation
  \[ 0 = E [\ell(-V(\bar{\xi} + k)) - z] \]
Liquidity Risk Measures (cont.)

Robust Representation

Let \( \Delta^N \) be the \( N \)-dimensional simplex in \( \mathbb{R}^{N+1} \).

Then \( \rho^V \) admits a robust representation

\[
\rho^V(\bar{\xi}) = - \min_{\bar{v} \in \Delta^N} (\bar{v} \cdot \bar{\xi} - \beta(\bar{v})) , \quad \bar{\xi} \in \mathbb{R}^{N+1},
\]

with penalty function

\[
\beta(\bar{v}) = - \inf_{V(\bar{\xi}) \in \mathcal{A}} \bar{v} \cdot \bar{\xi}, \quad \bar{v} \in \Delta^N.
\]
Numerical Case Studies
Market and Portfolio

- Portfolio vector
  \[ \bar{\xi} = (\xi_0, \xi_1, \xi_2) = (0, -3, 4) \]

- Exponential marginal demand-supply curves
  \[ m(x) = a \cdot e^{-bx} \]
  with \( a - 20 \sim \mathcal{LN}(1/2, 1/2) \)

- Liquidity constraints
  \[ \phi(\xi) = \alpha \cdot (\xi_1^- + \xi_2^-) \]

- Portfolio constraints
  \[ \mathcal{K} = [-4, \infty)^2 \]
## Liquidity-Adjusted Value

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<th>$\alpha$</th>
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<th>$\sigma^2$</th>
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Liquidity-Apdated Value (cont.)

Liquidity Adjusted Portfolio Values with $b = 0.5, \alpha = 5, 10, 15, 20$
### Liquidity-Adjusted Entropic Risk Measures

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<tr>
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<th>( b = 0.0005 )</th>
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<tr>
<td></td>
<td>( \alpha = 5 )</td>
<td>( \alpha = 10 )</td>
<td>( \alpha = 15 )</td>
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<td>( \beta = 0.02 )</td>
<td>( \frac{k_{RM}}{k_{PR}} )</td>
<td>( \frac{k_{RM}}{k_{PR}} )</td>
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<td>-21.7000</td>
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<td>( \alpha = 5 )</td>
<td>( \alpha = 10 )</td>
<td>( \alpha = 15 )</td>
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<tr>
<td>( \beta = 0.02 )</td>
<td>( \frac{k_{RM}}{k_{PR}} )</td>
<td>( \frac{k_{RM}}{k_{PR}} )</td>
<td>( \frac{k_{RM}}{k_{PR}} )</td>
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<tr>
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<td>3.6768/3.8298</td>
<td>11.7150/11.7200</td>
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<td>VaR(0.05)</td>
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<td>( \alpha = 5 )</td>
<td>( \alpha = 10 )</td>
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<tr>
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<td>0.7344/0.6064</td>
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<td>11.8280</td>
<td>31.4780</td>
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Conclusion
Conclusion

(i) Portfolio values and static risk measures can be adjusted for liquidity risk

(ii) Liquidity-adjusted risk measures are risk measures!

(iii) Numerical case studies clearly demonstrate liquidity effects

(iv) Further research: dynamic liquidity-adjusted risk measures