Hardness and Approximation Results for $L_p$–Ball Constrained Homogeneous Polynomial Optimization Problems

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In this paper, we establish hardness and approximation results for various $L_p$–ball constrained homogeneous polynomial optimization problems, where $p \in [2, \infty]$. Specifically, we prove that for any given $d \geq 3$ and $p \in [2, \infty]$, both the problem of optimizing a degree–$d$ homogeneous polynomial over the $L_p$–ball and the problem of optimizing a degree–$d$ multilinear form (regardless of its super–symmetry) over $L_p$–balls are NP–hard. On the other hand, we show that these problems can be approximated to within a factor of $\Omega\left(\frac{(\log n)^{(d-2)/p}}{n^{d/2-1}}\right)$ in deterministic polynomial time, where $n$ is the number of variables. We further show that with the help of randomization, the approximation guarantee can be improved to $\Omega\left(\frac{(\log n)^{d/2-1}}{n^{d/2-1}}\right)$, which is independent of $p$ and is currently the best for the aforementioned problems. Our results unify and generalize those in the literature, which focus either on the quadratic case or the case where $p \in \{2, \infty\}$. We believe that the wide array of tools used in this paper will have further applications in the study of polynomial optimization problems.

Key words: polynomial optimization; approximation algorithms; algorithmic convex geometry; convex programming

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1. Introduction. Motivated by its diverse applications and profound connections to various branches of mathematics, polynomial optimization has been the focus of much research effort during the past decade or so. From an algorithmic perspective, polynomial optimization problems are generally intractable. Thus, a fundamental research issue is to determine their approximability. One important class of problems whose approximability has been extensively investigated in recent years is that of homogeneous polynomial optimization with $L_2$–norm constraints. The first results in this direction were obtained by de Klerk et al. [9] and Barvinok [4], who showed that certain specially structured $L_2$–sphere constrained polynomial optimization problems admit polynomial–time approximation schemes (PTASes). These were then followed by the work of Luo and Zhang [21], in which an approximation algorithm was developed for homogeneous quartic optimization problems with quadratic constraints (which includes the $L_2$–ball as a special case). Around the same time, Ling et al. [19] considered the problem of approximately optimizing a biquadratic function over the Cartesian product of two $L_2$–spheres; while Zhang et al. [33] studied the hardness and approximability of certain $L_2$–sphere constrained homogeneous cubic optimization problems. Since then, there have been significant activities in this line of research. For instance, in [32, 20, 31], various researchers derived approximation results for the problem of optimizing a biquadratic function over quadratic constraints, thereby extending the results in [19]. In [12], He et al. improved