REMARKS ON THE QUANTUM DE FINETTI THEOREM FOR
BOSONIC SYSTEMS

MATHIEU LEWIN, PHAN THÀNH NAM, AND NICOLAS ROUGERIE

Abstract. The quantum de Finetti theorem asserts that the $k$-body density matrices of a $N$-body bosonic state approach a convex combination of Hartree states (pure tensor powers) when $N$ is large and $k$ fixed. In this note we review a construction due to Christandl, Mitchison, König and Renner [8] valid for finite dimensional Hilbert spaces, which gives a quantitative version of the theorem. We first propose a variant of their proof that leads to a slightly improved estimate. Next we provide an alternative proof of an explicit formula due to Chiribella [7], which gives the density matrices of the constructed state as a function of those of the original state.

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1. Introduction

Consider a system of $N$ bosons with one-particle state space $\mathcal{H}$, a separable Hilbert space. Pure states of this system are rank-one projectors $|\Psi\rangle\langle\Psi|$ onto normalized vectors $\Psi \in \mathcal{H}^N$, the symmetric tensor product of $N$ copies of $\mathcal{H}$. Mixed states are then convex combinations of pure states, that is, self-adjoint positive trace-class operators with trace 1:

$$S(\mathcal{H}^N) := \{\Gamma \in \mathcal{S}_1(\mathcal{H}^N), \Gamma = \Gamma^*, \Gamma \geq 0, \text{Tr}\Gamma = 1\}.$$

It is a well-known fact that Hartree states, i.e. projections onto tensor powers $\Psi = u^\otimes N \in \mathcal{H}^N$ (for $u$ a normalized vector in $\mathcal{H}$), play a very special role in the physics of bosonic systems. Indeed, since bosons, contrarily to fermions, do not satisfy the Pauli exclusion principle, there is a possibility for many particles to occupy the same quantum state, which is the meaning of the ansatz $u^\otimes N$. It is in fact the case for non interacting particles in the ground state of a one-body Hamiltonian, or in the thermal state at low enough temperature. This is the famous Bose-Einstein condensation phenomenon.

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