Van Lambalgen’s theorem fails for some computable measure

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Abstract

Van Lambalgen’s theorem states that a pair \((\alpha, \beta)\) of bitsequences is Martin-Löf random if and only if \(\alpha\) is Martin-Löf random and \(\beta\) is Martin-Löf random relative to \(\alpha\). The definition of Martin-Löf randomness can be generalized to define randomness relative to any computable measure \(P\). In [Information and Computation 209.2 (2011): 183-197, Theorem 3.3], Hayato Takahashi generalized van Lambalgen’s theorem for computable measures \(P\) on a product of two Cantor spaces; he showed that the equivalence holds for each \(\beta\) for which the conditional probability \(P(\cdot|\beta)\) is computable. In this note we show that this additional condition on \(\beta\) can not be removed by providing a measure for which van Lambalgen’s theorem fails.

Michiel van Lambalgen characterized Martin-Löf randomness of a pair of sequences:

Theorem 1 ([4]). The following are equivalent for a pair \((\alpha, \beta)\) of sequences:

• \((\alpha, \beta)\) is Martin-Löf random,
• \(\alpha\) is Martin-Löf random and \(\beta\) is Martin-Löf random relative to \(\alpha\).

One can replace Lebesgue measure in the definition of Martin-Löf randomness, by any other computable measure \(P\). We call sequences that are random in this sense \(P\)-random. Similar for a pair of measures. To generalize the theorem above for computable bivariate measures \(P\), we need to fix the conditional measure carefully.\(^1\) Let \(2^\omega\) denote Cantor space. For any string \(x\), let \([x]\) be the (basic open) set containing all extensions of \(x\). Following Takahashi [6], we define for each measurable set \(S \subset 2^\omega\):

\[
P_C(S|\beta) = \lim_{n \to \infty} \frac{P(S \times [\beta_1 \ldots \beta_n])}{P(2^\omega \times [\beta_1 \ldots \beta_n])}.
\]

Let the marginal distribution be \(P_M(S) = P(2^\omega \times S)\). In [6], Hayato Takahashi proved that if \(\beta\) is \(P_M\)-random, then \(P_C(\cdot|\beta)\) is defined on all measurable sets.

In [1] it was shown that for computable \(P\) and for some fixed \(\beta\) the measure \(P_C(\cdot|\beta)\) might not be computable relative to \(\beta\), (albeit, using a different and related definition of computability; it can even hold for a \(P_M\)-measure one set of \(\beta\)). Hayato Takahashi generalized van Lambalgen’s theorem as follows:

Theorem 2 ([7, 8]). For any computable bivariate measure \(P\) and any \(\beta\) such that \(P_C(\cdot|\beta)\) is computable relatively to \(\beta\), the following are equivalent:

• \((\alpha, \beta)\) is \(P\)-Martin-Löf random,
• \(\beta\) is \(P_M\)-random and \(\alpha\) is \(P_C(\cdot|\beta)\)-random relative to \(\beta\).

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\(^1\)Conditional measure is typically defined by the Radon-Nikodym derivative of the bivariate measure to the marginal measure, but this derivative is only defined up to a measure null set. We fix such a derivative.