PAIRS OF PANTS, POCHHAMMER CURVES AND $L^2$-INVARIANTS

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Abstract. We propose an intuitive interpretation for nontrivial $L^2$-Betti numbers of compact Riemann surfaces in terms of certain loops in embedded pairs of pants. This description uses twisted homology associated to the Hurewicz map of the surface, and it satisfies a sewing property with respect to a large class of pair-of-pants decompositions. Applications to supersymmetric quantum mechanics incorporating Aharonov–Bohm phases are briefly discussed, for both point particles and topological solitons (Abelian and non-Abelian vortices) in two dimensions.

1. Introduction

The theory of $L^2$-invariants was created out of an effort by Atiyah to extend the index theorem for elliptic operators on compact manifolds to certain noncompact situations. In a ground-breaking paper [1], he proposed a $\Gamma$-index for elliptic differential operators on an infinite Galois cover $\tilde{M}$ of a compact manifold $M$ which combined analysis of operators over a fundamental domain for the action of the group of deck transformations $\Gamma$, identified with an open dense subset of $M$, with analysis on the Hilbert space of sequences $\ell^2(\Gamma)$. This machinery, applied to the operator $d + d^*$ on the manifold $\tilde{M}$ with a cocompact $\Gamma$-invariant Riemannian metric, leads to invariants $b^{(2)}_i$ or $b_i^{(2)}$ which generalize the rank of $L^2$-cohomology groups of finite covers; it can be used to great advantage in cases where the $L^2$-cohomology of the covering space is an infinitely generated vector space. These $L^2$-Betti numbers are obtained as Murray–von Neumann (or renormalized) dimensions [22] of certain Hilbert $\mathcal{N}(\Gamma)$-modules, where $\mathcal{N}(\Gamma) := \mathcal{B}(\ell^2(\Gamma))^\Gamma$ is the von Neumann algebra associated to the discrete group $\Gamma$. For example, if $M = \mathbb{T}^n$ is an $n$-torus (the setting for classical Fourier analysis), applying this construction to the universal cover $\tilde{M} = \mathbb{R}^n$ leads to all $L^2$-Betti numbers $b^{(2)}_i(\mathbb{R}^n, \Gamma)$ for $\Gamma = \pi_1(\mathbb{T}^n) \cong \mathbb{Z}^n$ being zero — this follows from multiplicity under finite covers and a Künneth-type formula. Quite often, $L^2$-Betti numbers vanish in situations where the ordinary Betti numbers of the quotient space do not, and they are not necessarily integral real numbers; a question that was open until recently was whether they can ever be irrational. For a layout of the theory of $L^2$-invariants, and in order to appreciate the impact that these ideas have had in mathematics so far, we refer the reader to the textbook [19]. An informal account of some basic notions, tailored to the purposes of the present paper, will be given in section 4 below.

An example leading to nontrivial $L^2$-invariants is provided by the universal cover $\mathbb{D}$ (the disc) of a compact oriented Riemann surface $M = \Sigma$ of genus $g > 1$. This simple but somewhat crucial example was already considered in Atiyah’s original paper, where he illustrated his new theory by showing that

\[
\begin{align*}
b^{(2)}_i(\Sigma, \Gamma) &= \begin{cases} 0 & \text{if } i \neq 1, \\ 2g - 2 & \text{if } i = 1. \end{cases}
\end{align*}
\]

Availing oneself of the standard machinery [19], the argument runs along the following lines. The $L^2$-Betti numbers in degrees $i = 0$ or $2$ must vanish because $\mathbb{D}$ has no compact component, thus any nontrivial $L^2$-Betti numbers must lie in degree 1. Then one uses that