An Operational Characterization of the Notion of Probability by Algorithmic Randomness

Kohtaro Tadaki*

Abstract—The notion of probability plays an important role in almost all areas of science. In modern mathematics, however, probability theory means nothing other than measure theory, and an operational characterization of the notion of probability is not established yet. In this paper, based on the toolkit of algorithmic randomness we present an operational characterization of the notion of probability in the case where the sample space of the underlying probability space is finite.

Keywords—probability, algorithmic randomness, operational characterization, Martin-Löf randomness, Bernoulli measure

1 Introduction

The notion of probability plays an important role in almost all areas of science. In modern mathematics, however, probability theory means nothing other than measure theory, and an operational characterization of the notion of probability is not established yet.

In the past century, however, there was a comprehensive attempt to provide such a characterization. Namely, von Mises developed a mathematical theory of repetitive events which is aimed at reformulating the theory of probability and statistics based on an operational characterization of the notion of probability [14, 15]. In a series of comprehensive works which began in 1919, von Mises developed this theory and, in particular, introduced the notion of collective as a mathematical idealization of a long sequence of outcomes of experiments or observations repeated under a set of invariable conditions, such as the repeated tossing of a coin or of a pair of dice.

The collective plays a role as an operational characterization of the notion of probability, and is an infinite sequence of sample points in the sample space of a probability space. As the randomness property of the collective, von Mises assumes that all “reasonable” infinite subsequences of a collective satisfy the law of large numbers with the identical limit value, where the subsequences are selected using “acceptable selection rules.” Wald [16, 17] later showed that for any countable collection of selection rules, there are sequences that are collectives in the sense of von Mises, but at the time it was unclear exactly what types of selection rules should be acceptable. There seemed to von Mises to be no canonical choice.

Later, with the development of computability theory and the introduction of generally accepted precise mathematical definitions of the notions of algorithm and computable function, Church [7] made the first explicit connection between computability theory and randomness by suggesting that a selection rule be considered acceptable if and only if it is computable. In 1939, however, Ville [13] revealed the defect of the notion of collective. Namely, he showed that for any countable collection of selection rules, there is a sequence that is random in the sense of von Mises but has properties that make it clearly nonrandom. (For the development of the theory of collectives from the point of view of the definition of randomness, see Downey and Hirschfeldt [8].)

In 1966, Martin-Löf [9] introduced the definition of random sequences, which is called Martin-Löf randomness nowadays, and plays a central role in the recent development of algorithmic randomness. At the same time, he introduced the notion of Martin-Löf randomness with respect to Bernoulli measure [9]. He then pointed out that this notion overcomes the defect of collective, and this can be regarded precisely as the collective which von Mises wanted to define. However, he did not develop probability theory based on Martin-Löf random sequence with respect to Bernoulli measure.

Algorithmic randomness is a field of mathematics which studies the definitions of random sequences and their property [10, 8]. However, the research on algorithmic randomness would seem only interested in the notions of randomness and their property, and not seem to have tried to develop probability theory based Martin-Löf randomness with respect to Bernoulli measure in an operational manner so far.

The subject of this paper is to make such an attempt. Namely, in this paper we present an operational characterization of the notion of probability based on Martin-Löf randomness with respect to Bernoulli measure. As the first step of the research of this line, we only consider the case of finite probability space, i.e., the case where the sample space of the underlying probability space is finite, for simplicity. The investigation of the case of general probability spaces is left to the future study. We emphasize that the Bernoulli measure which we consider in this paper is not required to be computable at all, while the measures considered in algorithmic randomness are usually computable. Thus, the results in this paper hold for any finite probability space.

Due to the 6-page limit, we omit some proofs. A full paper which describes all the proofs and other related results is in preparation.

* Research and Development Initiative, Chuo University, 1-13-27 Kasuga, Bunkyo-ku, Tokyo 112-8551, Japan. E-mail: tadaki@kc.chuo-u.ac.jp WWW: http://www2.odn.ne.jp/tadaki/